Stochastic Discrete Hamiltonian Variational Integrators

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Stochastic Hamiltonian systems

- Mechanical systems:
  - subject to random perturbations
  - whose parameters are not precisely known
  - stochastic landmarks, dissipation phenomena, particle storage rings

$$dq_a = \frac{\partial H}{\partial p_a} dt + \sum_{i=1}^{M} \frac{\partial h_i}{\partial p_a} \circ dW_t^i$$

$$dp_a = -\frac{\partial H}{\partial q_a} dt - \sum_{i=1}^{M} \frac{\partial h_i}{\partial q_a} \circ dW_t^i$$

Plan

1. Geometric and variational integration
2. Stochastic variational principle in phase space
3. Stochastic Galerkin variational integrators
4. Examples and numerical tests

Why geometric integration?

- Preserving the geometric properties of the flow of a differential equation
- Symplectic and variational integrators
- Long-time integration – excellent behavior (provable by ***backward error analysis***)
- Applications in astronomy, molecular dynamics, mechanics, theoretical physics, etc.

Hamiltonian systems

- Phase space T^*Q with coordinates (q^μ, p_μ)
- Hamiltonian

$$H : T^*Q \longrightarrow \mathbb{R}$$

- Hamiltonian equations

$$\dot{q}^\mu = \frac{\partial H}{\partial p_\mu}, \qquad \dot{p}_\mu = -\frac{\partial H}{\partial q^\mu}.$$

Hamiltonian systems

- Symplectic form

$$\Omega = -d\Theta = dq^{\mu} \wedge dp_{\mu}$$

- Symplecticity of the flow

$$(F_t^H)^*\Omega = \Omega$$

- Conservation of energy

$$H \circ F_t^H = H$$

Symplectic integrators

- Numerical scheme

$$F_h : T^*Q \longrightarrow T^*Q$$

$$(q_{k+1}, p_{k+1}) = F_h(q_k, p_k)$$

- Symplectic integrator

$$(DF_h)^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} DF_h = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Symplectic integrators

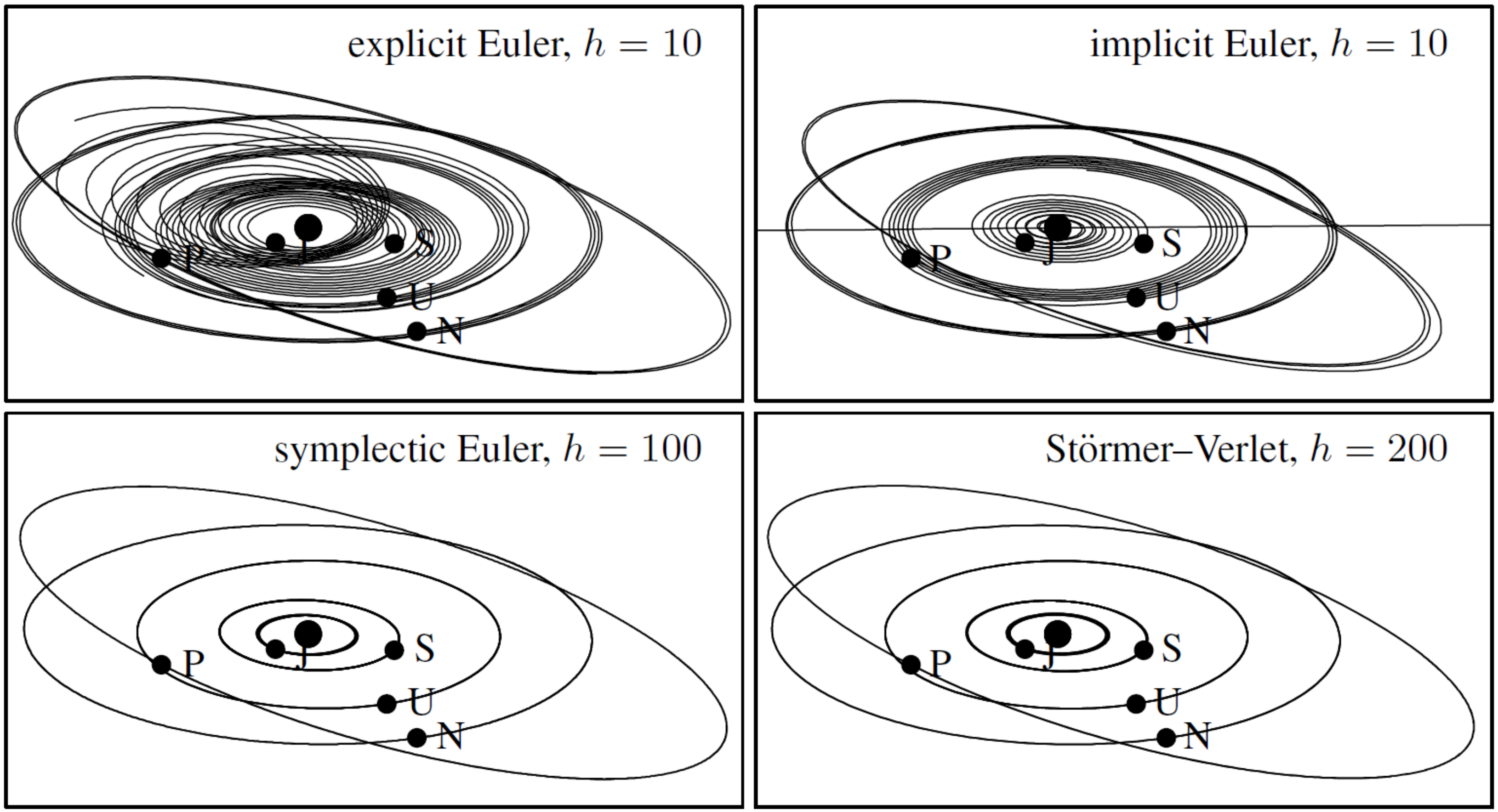
- Symplectic Euler scheme

$$q_{k+1} = q_k + h\frac{\partial H}{\partial p}(q_{k+1}, p_k),$$
$$p_{k+1} = p_k - h\frac{\partial H}{\partial q}(q_{k+1}, p_k).$$

- Backward error analysis

$$\tilde{H}(q,p) = H(q,p) + hH_2(q,p) + h^2 H_3(q,p) + \ldots$$

Example: Outer Solar System



(Hairer, Lubich, Wanner, 2002)

Lagrangian systems

- Configuration space TQ with coord. $(q^\mu, \dot{q}^\mu)$

- Lagrangian

$$L : TQ \longrightarrow \mathbb{R}$$

- Action functional

$$S[q(t)] = \int_a^b L\big(q^\mu(t), \dot{q}^\mu(t)\big)\, dt$$

Lagrangian systems

- Hamilton's principle

$$dS[q(t)] \cdot \delta q(t) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} S[q_\epsilon(t)] = 0$$

- Euler-Lagrange equations

$$\frac{\partial L}{\partial q^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\mu} = 0$$

- Symplectic structure on TQ

Variational integrators

- Discrete state space $Q \times Q$ with coord. $(q^{\mu}, \bar{q}^{\mu})$

- Discrete Lagrangian

$$L_d : Q \times Q \longrightarrow \mathbb{R}$$

- Discrete action for a discrete path $(q_0, q_1, \ldots, q_N)$

$$S = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$$

Variational integrators

- Discrete Euler-Lagrange equations

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) = 0$$

$$F_{L_d} : Q \times Q \longrightarrow Q \times Q \qquad F_{L_d}(q_{k-1}, q_k) = (q_k, q_{k+1})$$

- Position-momentum formulation

$$p_k = -D_1 L_d(q_k, q_{k+1}),$$

$$p_{k+1} = D_2 L_d(q_k, q_{k+1}).$$

$$\tilde{F}_{L_d} : T^*Q \longrightarrow T^*Q$$

Variational integrators

- Variational integrators are symplectic
- Discrete Noether's theorem
- Example: symplectic Euler scheme

$$L_d(q,\bar{q}) = hL\left(\bar{q}, \frac{\bar{q}-q}{h}\right)$$

- *(Marsden & West, 2001)*

Stochastic Hamiltonian systems

$$dq = \frac{\partial H}{\partial p} dt + \frac{\partial h}{\partial p} \circ dW(t)$$

$$dp = -\frac{\partial H}{\partial q} dt - \frac{\partial h}{\partial q} \circ dW(t)$$

- Assumptions:

- $Q \cong \mathbb{R}^N, \qquad T^*Q = Q \times Q^* \cong \mathbb{R}^N \times \mathbb{R}^N, \qquad TQ = Q \times Q \cong \mathbb{R}^N \times \mathbb{R}^N$
- $H : T^*Q \longrightarrow \mathbb{R}, \qquad h : T^*Q \longrightarrow \mathbb{R}$
- $(\Omega, \mathcal{F}, \mathbb{P}), \qquad \{\mathcal{F}_t\}_{t \geq 0}$

Stochastic symplectic flow

- Pathwise unique stochastic flow

$$F_{t,t_0} : \Omega \times T^*Q \longrightarrow T^*Q$$

- Mean-square differentiable and almost surely a diffeomorphism

- Symplectic

$$F_{t,t_0}^* \Omega_{T^*Q} = \Omega_{T^*Q}$$

where $\Omega_{T^*Q} = \sum_{i=1}^{N} dq^i \wedge dp^i$

Stochastic variational principle

- Function vector space

$$C([t_a,t_b]) = \left\{(q,p) : \Omega \times [t_a,t_b] \longrightarrow T^*Q \,\middle|\, q,\ p \text{ a.s. cont. } \mathcal{F}_t\text{-adapted semimartingales}\right\}$$

- Action functional $\mathcal{B} : \Omega \times C([t_a,t_b]) \longrightarrow \mathbb{R}$

$$\mathcal{B}[q(\cdot),p(\cdot)] = p(t_b)q(t_b) - \int_{t_a}^{t_b} \left[p \circ dq - H\big(q(t),p(t)\big)\,dt - h\big(q(t),p(t)\big) \circ dW(t)\right]$$

Stochastic variational principle

- Noise in the action functional (for Lagrangian systems)
  - *Bismut 1982*
  - *Bou-Rabee & Owhadi 2009*

- Zero noise limit
  - *Leok & Zhang 2011*

Stochastic variational principle

Theorem 1 (Stochastic Variational Principle in Phase Space). *Suppose that* $H(q,p)$ *and* $h(q,p)$ *are* C^2 *functions of their arguments with globally Lipschitz derivatives. If the curve* $\big(q(t),p(t)\big)$ *in* T^*Q *satisfies the stochastic Hamiltonian system for* $t \in [t_a, t_b]$, *where* $t_b \geq t_a > 0$, *then the pair* $\big(q(\cdot),p(\cdot)\big)$ *is a critical point of the stochastic action functional, that is,*

$$\delta\mathcal{B}\big[q(\cdot),p(\cdot)\big] \equiv \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{B}\big[q(\cdot)+\epsilon\delta q(\cdot), p(\cdot)+\epsilon\delta p(\cdot)\big] = 0$$

almost surely for all variations $\big(\delta q(\cdot),\delta p(\cdot)\big) \in C([t_a,t_b])$ *such that almost surely* $\delta q(t_a)=0$ *and* $\delta p(t_b)=0$.

Stochastic variational principle

- Sketch of proof

- $$\int_{t_a}^{t_b} p(t) \circ d\delta q(t) = p(t_b)\delta q(t_b) - p(t_a)\delta q(t_a) - \int_{t_a}^{t_b} \delta q(t) \circ dp(t)$$

- $$\begin{aligned} \delta\mathcal{B}[q(\cdot), p(\cdot)] = \int_{t_a}^{t_b} \delta q(t)\Big[\circ dp(t) + \frac{\partial H}{\partial q}\big(q(t), p(t)\big)\, dt + \frac{\partial h}{\partial q}\big(q(t), p(t)\big) \circ dW(t) \Big] \\ - \int_{t_a}^{t_b} \delta p(t)\Big[\circ dq(t) - \frac{\partial H}{\partial p}\big(q(t), p(t)\big)\, dt - \frac{\partial h}{\partial p}\big(q(t), p(t)\big) \circ dW(t) \Big] \end{aligned}$$

Stochastic variational principle

- Converse theorem
  - proved in case $h = h(q)$ for Lagrangian systems *(Bou-Rabee & Owhadi, 2009)*
  - may not be true in general *(Lazaro-Cami & Ortega, 2008)*

Stochastic type-II generating function

- Generating function

$$S(q_a, p_b) = \mathcal{B}\left[\bar{q}(\cdot; q_a, p_b), \bar{p}(\cdot; q_a, p_b)\right]$$

where $(\bar{q}, \bar{p})$ is the exact solution such that

$$\bar{q}(t_a; q_a, p_b) = q_a, \ \bar{p}(t_b; q_a, p_b) = p_b$$

Stochastic type-II generating function

Theorem 2. *The function* $S(q_a, p_b)$ *is a type-II stochastic generating function for the stochastic mapping* F_{t_b,t_a}*, that is,* $F_{t_b,t_a} : (q_a, p_a) \longrightarrow (q_b, p_b)$ *is implicitly given by the equations*

$$q_b = D_2 S(q_a, p_b), \qquad\qquad p_a = D_1 S(q_a, p_b),$$

where the derivatives are understood in the mean-square sense.

Stochastic type-II generating function

- Sketch of proof

$$\frac{\partial S}{\partial q_a}(q_a, p_b) = \bar{p}(t_a) + \int_{t_a}^{t_b} \frac{\partial \bar{q}(t)}{\partial q_a}\left[\circ\, d\bar{p} + \frac{\partial H}{\partial q}\big(\bar{q}(t), \bar{p}(t)\big)\, dt + \frac{\partial h}{\partial q}\big(\bar{q}(t), \bar{p}(t)\big) \circ dW(t) \right]$$
$$+ \int_{t_a}^{t_b} \frac{\partial \bar{p}(t)}{\partial q_a}\left[\circ\, d\bar{q} - \frac{\partial H}{\partial p}\big(\bar{q}(t), \bar{p}(t)\big)\, dt - \frac{\partial h}{\partial p}\big(\bar{q}(t), \bar{p}(t)\big) \circ dW(t) \right] = \bar{p}(t_a)$$

Stochastic Noether's theorem

Theorem 3 (Stochastic Noether's theorem). *Suppose that the Hamiltonians* $H : T^*Q \longrightarrow \mathbb{R}$ *and* $h : T^*Q \longrightarrow \mathbb{R}$ *are invariant with respect to the cotangent lift action* $\Phi^{T^*Q} : G \times T^*Q \longrightarrow T^*Q$ *of the Lie group* G*, that is,*

$$H \circ \Phi_g^{T^*Q} = H, \qquad h \circ \Phi_g^{T^*Q} = h,$$

for all $g \in G$*. Then the cotangent lift momentum map* $J : T^*Q \longrightarrow \mathfrak{g}^*$ *associated with* Φ^{T^*Q}*, in coordinates given by*

$$J_\xi(q,p) = p \cdot \xi_Q(q),$$

is almost surely preserved along the solutions of the stochastic Hamiltonian system.

Stochastic Galerkin Variational Integrator

- Generating function

$$S(q_a, p_b) = \underset{\substack{(q,p)\in C([t_a,t_b]) \\ q(t_a)=q_a,\ p(t_b)=p_b}}{\operatorname{ext}} \mathcal{B}[q(\cdot), p(\cdot)]$$

- Step 1 – extremize over a subspace
- Step 2 – approximate integrals with quadrature rules

Stochastic Galerkin Variational Integrator

- Discrete set of times

$$t_k = k \cdot \Delta t, \quad k = 0, 1, \ldots, K, \quad \Delta t = T/K$$

- Discrete curve $\{(q_k, p_k)\}_{k=0,\ldots,K}$

$$(q_{k+1}, p_{k+1}) \approx F_{t_{k+1}, t_k}(q_k, p_k)$$

Stochastic Galerkin Variational Integrator

- Approximation space

$$C^s([t_k, t_{k+1}]) = \left\{(q,p) \in C([t_k, t_{k+1}]) \,\middle|\, q \text{ is a polynomial of degree } s\right\}$$

- Lagrange polynomials for $0 = d_0 < d_1 < \ldots < d_s = 1$

$$q_d(t_k + \tau\Delta t; q^\mu) = \sum_{\mu=0}^{s} q^\mu l_{\mu,s}(\tau), \qquad \dot{q}_d(t_k + \tau\Delta t; q^\mu) = \frac{1}{\Delta t}\sum_{\mu=0}^{s} q^\mu \dot{l}_{\mu,s}(\tau)$$

- Quadrature rules

$$(\alpha_i, c_i)_{i=1}^r, \quad (\beta_i, c_i)_{i=1}^r, \quad 0 \le c_1 < \ldots < c_r \le 1$$

Stochastic Galerkin Variational Integrator

- Discrete stochastic Hamiltonian

$$H_d^+(q_k, p_{k+1}) = \underset{\substack{q^1,\ldots,q^s \in Q \\ P_1,\ldots,P_r \in Q^* \\ q^0 = q_k}}{\text{ext}} \Big\{ p_{k+1} q^s - \Delta t \sum_{i=1}^{r} \alpha_i \Big[P_i \dot{q}_d(t_k + c_i \Delta t) - H\big(q_d(t_k + c_i \Delta t), P_i\big) \Big] + \Delta W \sum_{i=1}^{r} \beta_i h\big(q_d(t_k + c_i \Delta t), P_i\big) \Big\}$$

where $P_i \equiv p(t_k + c_i \Delta t)$

- Discrete flow

$$q_{k+1} = D_2 H_d^+(q_k, p_{k+1}), \qquad p_k = D_1 H_d^+(q_k, p_{k+1})$$

Stochastic Galerkin Variational Integrator

- System defining the integrator

$$
\begin{aligned}
-p_k &= \sum_{i=1}^{r} \alpha_i \Big[P_i \dot{l}_{0,s}(c_i) - \Delta t \frac{\partial H}{\partial q}\big(t_k + c_i \Delta t\big) l_{0,s}(c_i) \Big] - \Delta W \sum_{i=1}^{r} \beta_i \frac{\partial h}{\partial q}\big(t_k + c_i \Delta t\big) l_{0,s}(c_i) \\
0 &= \sum_{i=1}^{r} \alpha_i \Big[P_i \dot{l}_{\mu,s}(c_i) - \Delta t \frac{\partial H}{\partial q}\big(t_k + c_i \Delta t\big) l_{\mu,s}(c_i) \Big] - \Delta W \sum_{i=1}^{r} \beta_i \frac{\partial h}{\partial q}\big(t_k + c_i \Delta t\big) l_{\mu,s}(c_i) \\
p_{k+1} &= \sum_{i=1}^{r} \alpha_i \Big[P_i \dot{l}_{s,s}(c_i) - \Delta t \frac{\partial H}{\partial q}\big(t_k + c_i \Delta t\big) l_{s,s}(c_i) \Big] - \Delta W \sum_{i=1}^{r} \beta_i \frac{\partial h}{\partial q}\big(t_k + c_i \Delta t\big) l_{s,s}(c_i) \\
\alpha_i \dot{q}_d(t_k + c_i \Delta t) &= \alpha_i \frac{\partial H}{\partial p}\big(t_k + c_i \Delta t\big) + \beta_i \frac{\Delta W}{\Delta t} \frac{\partial h}{\partial p}\big(t_k + c_i \Delta t\big) \\
q_{k+1} &= q^s
\end{aligned}
$$

where $H(t_k + c_i \Delta t) \equiv H\big(q_d(t_k + c_i \Delta t), p(t_k + c_i \Delta t)\big)$

Discrete flow

- Symplecticity

$$\left(F^{+}_{t_{k+1},t_k}\right)^{*}\Omega_{T^*Q} = \Omega_{T^*Q}$$

- Proof:

$$0 = ddH^{+}(q_k, p_{k+1}) = \sum_{i=1}^{N} dq^i_{k+1} \wedge dp^i_{k+1} - \sum_{i=1}^{N} dq^i_k \wedge dp^i_k = (F^{+}_{t_{k+1},t_k})^{*}\Omega_{T^*Q} - \Omega_{T^*Q}$$

Discrete stochastic Noether's theorem

- Action of a Lie group

$$\Phi : G \times Q \longrightarrow Q$$

- Equivariance of the interpolating polynomial

$$\Phi_g^{TQ}\Big(q_d(t;q^\mu),\dot{q}_d(t;q^\mu)\Big) = \Big(q_d\big(t;\Phi_g(q^\mu)\big),\dot{q}_d\big(t;\Phi_g(q^\mu)\big)\Big)$$

Discrete stochastic Noether's theorem

Theorem 4 (Discrete stochastic Noether's theorem). *Suppose that the Hamiltonians* $H : T^*Q \longrightarrow \mathbb{R}$ *and* $h : T^*Q \longrightarrow \mathbb{R}$ *are invariant with respect to the cotangent lift action* $\Phi^{T^*Q} : G \times T^*Q \longrightarrow T^*Q$ *of the Lie group* G*, that is,*

$$H \circ \Phi_g^{T^*Q} = H, \qquad h \circ \Phi_g^{T^*Q} = h,$$

for all $g \in G$*, and suppose the interpolating polynomial* $q_d(t; q^\mu)$ *is equivariant with respect to* G*. Then the cotangent lift momentum map* J *associated with* Φ^{T^*Q} *is almost surely preserved, i.e., a.s.*

$$\boxed{J(q_{k+1}, p_{k+1}) = J(q_k, p_k).}$$

Example: Stochastic midpoint method

- Polynomials of degree $s = 1$
- Midpoint rule $r = 1,\ c_1 = 1/2,\ \alpha_1 = \beta_1 = 1$

$$q_{k+1} = q_k + \frac{\partial H}{\partial p}\left(\frac{q_k + q_{k+1}}{2}, \frac{p_k + p_{k+1}}{2}\right)\Delta t + \frac{\partial h}{\partial p}\left(\frac{q_k + q_{k+1}}{2}, \frac{p_k + p_{k+1}}{2}\right)\Delta W$$

$$p_{k+1} = p_k - \frac{\partial H}{\partial q}\left(\frac{q_k + q_{k+1}}{2}, \frac{p_k + p_{k+1}}{2}\right)\Delta t - \frac{\partial h}{\partial q}\left(\frac{q_k + q_{k+1}}{2}, \frac{p_k + p_{k+1}}{2}\right)\Delta W$$

- *(Milstein, Repin & Tretyakov, 2002)*

Example: Stochastic Störmer-Verlet method

- Polynomials of degree $s = 2$
- Trapezoidal rule
  $r = 2,\ c_1 = 0,\ c_2 = 1,\ \alpha_1 = \beta_1 = 1/2,\ \alpha_2 = \beta_2 = 1/2$

$$P_1 = p_k - \frac{1}{2}\frac{\partial H}{\partial q}(q_k, P_1)\Delta t - \frac{1}{2}\frac{\partial h}{\partial q}(q_k, P_1)\Delta W$$

$$q_{k+1} = q_k + \frac{1}{2}\frac{\partial H}{\partial p}(q_k, P_1)\Delta t + \frac{1}{2}\frac{\partial H}{\partial p}(q_{k+1}, P_1)\Delta t + \frac{1}{2}\frac{\partial h}{\partial p}(q_k, P_1)\Delta W + \frac{1}{2}\frac{\partial h}{\partial p}(q_{k+1}, P_1)\Delta W$$

$$p_{k+1} = P_1 - \frac{1}{2}\frac{\partial H}{\partial q}(q_{k+1}, P_1)\Delta t - \frac{1}{2}\frac{\partial h}{\partial q}(q_{k+1}, P_1)\Delta W$$

- *(Ma & Ding, 2015)*

Example: Stochastic trapezoidal method

- Polynomials of degree $s = 1$
- Trapezoidal rule

$r = 2,\ c_1 = 0,\ c_2 = 1,\ \alpha_1 = \beta_1 = 1/2,\ \alpha_2 = \beta_2 = 1/2$

$$p_k = \frac{1}{2}(P_1 + P_2) + \frac{1}{2}\frac{\partial H}{\partial q}(q_k, P_1)\Delta t + \frac{1}{2}\frac{\partial h}{\partial q}(q_k, P_1)\Delta W$$

$$p_{k+1} = \frac{1}{2}(P_1 + P_2) - \frac{1}{2}\frac{\partial H}{\partial q}(q_{k+1}, P_2)\Delta t - \frac{1}{2}\frac{\partial h}{\partial q}(q_{k+1}, P_2)\Delta W$$

$$q_{k+1} = q_k + \frac{\partial H}{\partial p}(q_k, P_1)\Delta t + \frac{\partial h}{\partial p}(q_k, P_1)\Delta W$$

$$q_{k+1} = q_k + \frac{\partial H}{\partial p}(q_{k+1}, P_2)\Delta t + \frac{\partial h}{\partial p}(q_{k+1}, P_2)\Delta W$$

Numerical tests: convergence

- Kubo oscillator

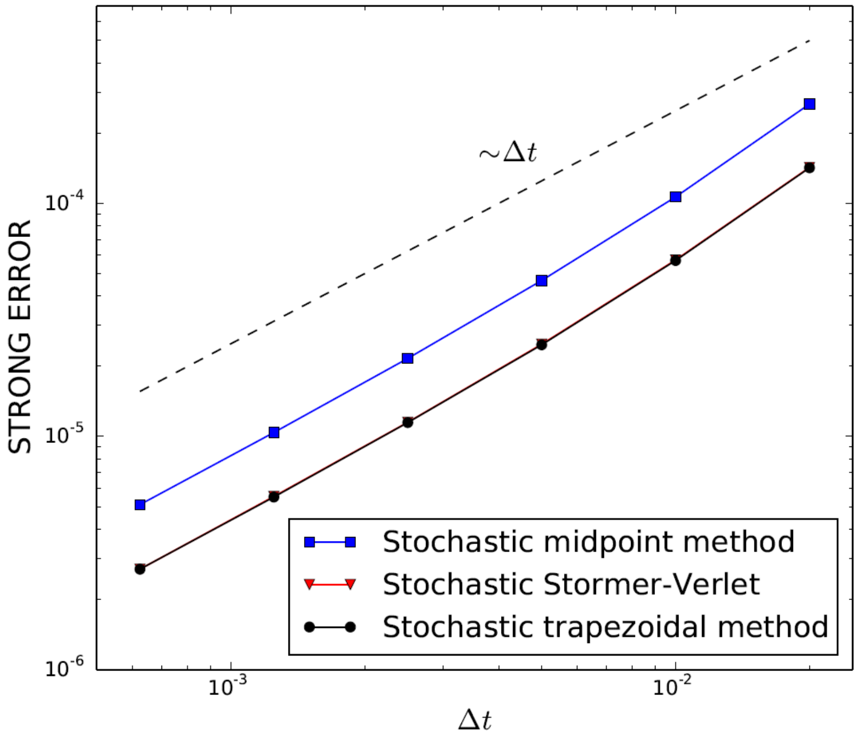
$$H(q,p) = p^2/2 + q^2/2 \qquad h(q,p) = \beta(p^2/2 + q^2/2)$$

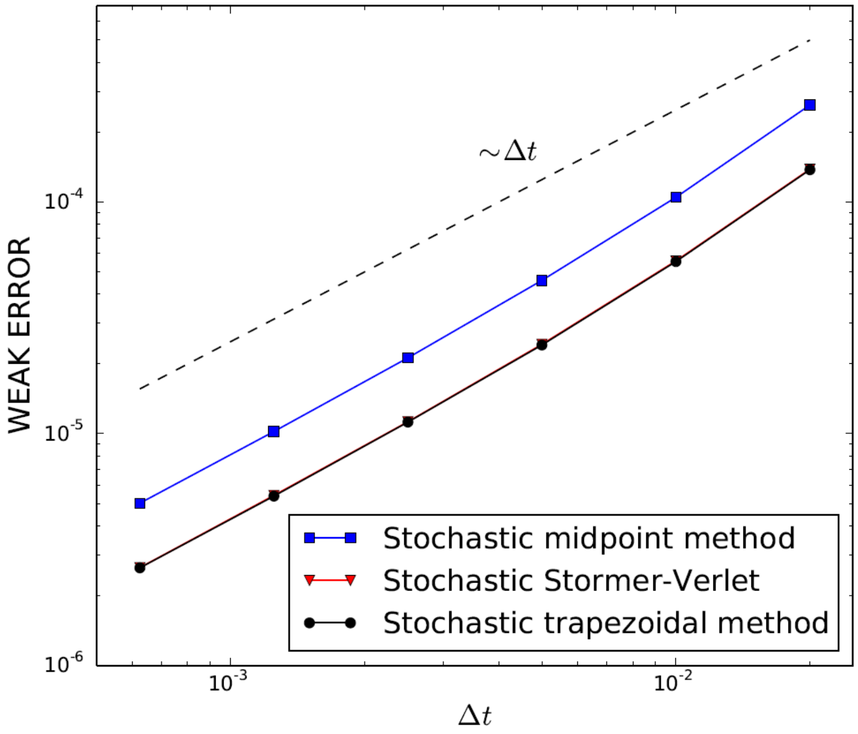
- Exact solution

$$q(t) = p_0 \sin(t + \beta W(t)) + q_0 \cos(t + \beta W(t))$$
$$p(t) = p_0 \cos(t + \beta W(t)) - q_0 \sin(t + \beta W(t))$$

- Hamiltonian is preserved

Numerical tests: convergence



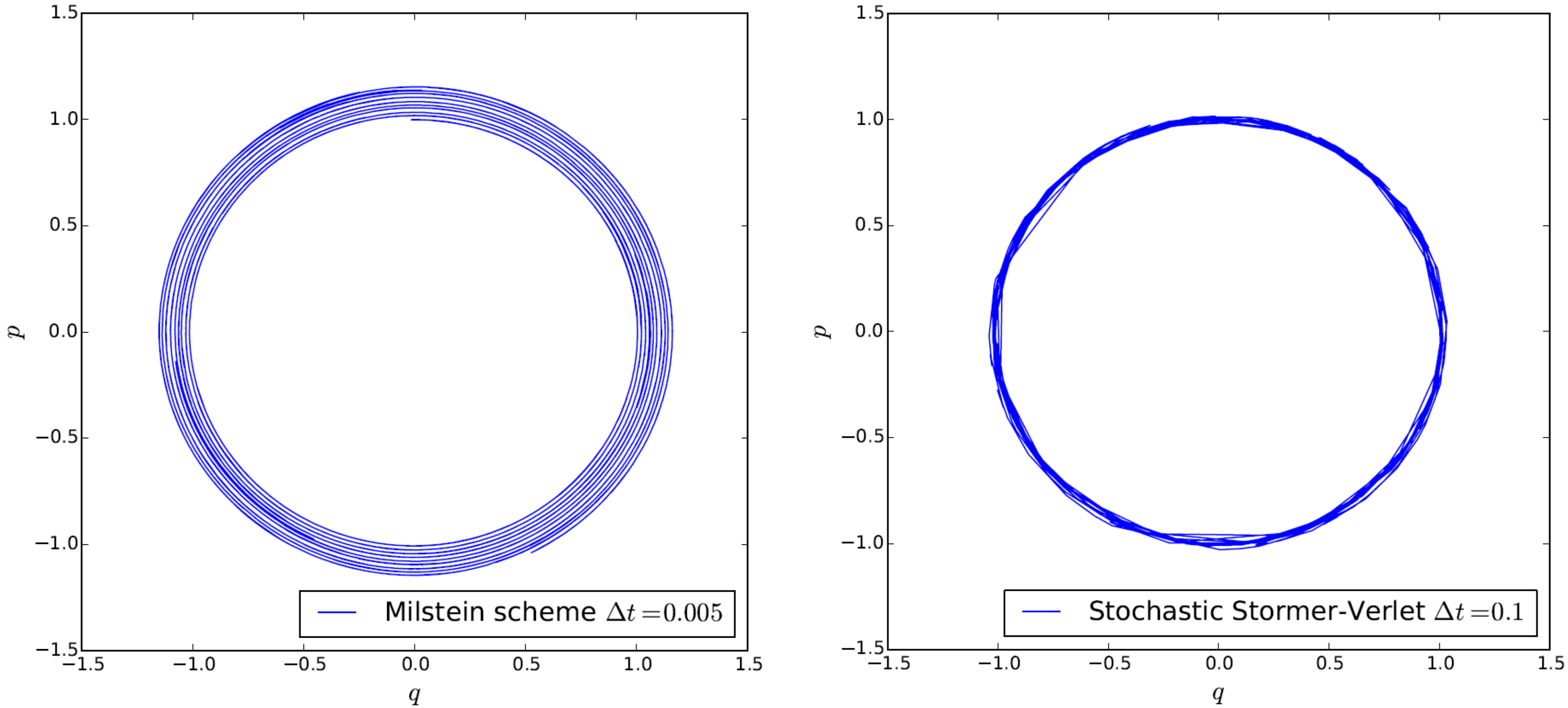


- Strong error $E(\|z_K - \bar{z}(T)\|)$
- Weak error $\|E(z_K) - E(\bar{z}(T))\|$

$$z = (q, p)$$

Numerical tests: Hamiltonian behavior

- Kubo oscillator $0 \le t \le 60$



Numerical tests: Hamiltonian behavior

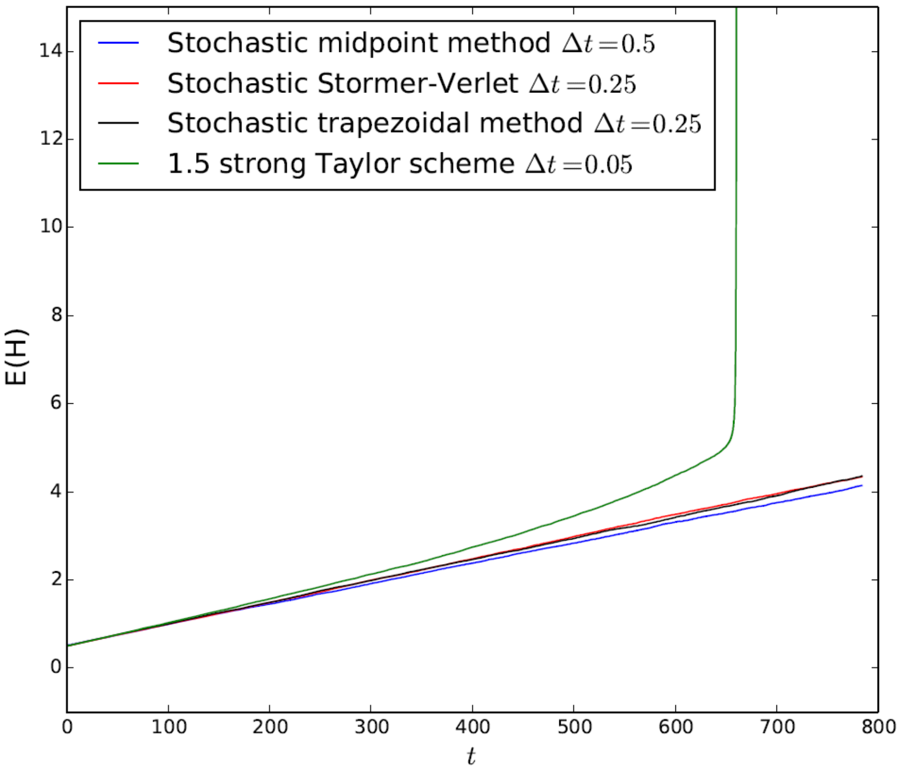
- Anharmonic oscillator

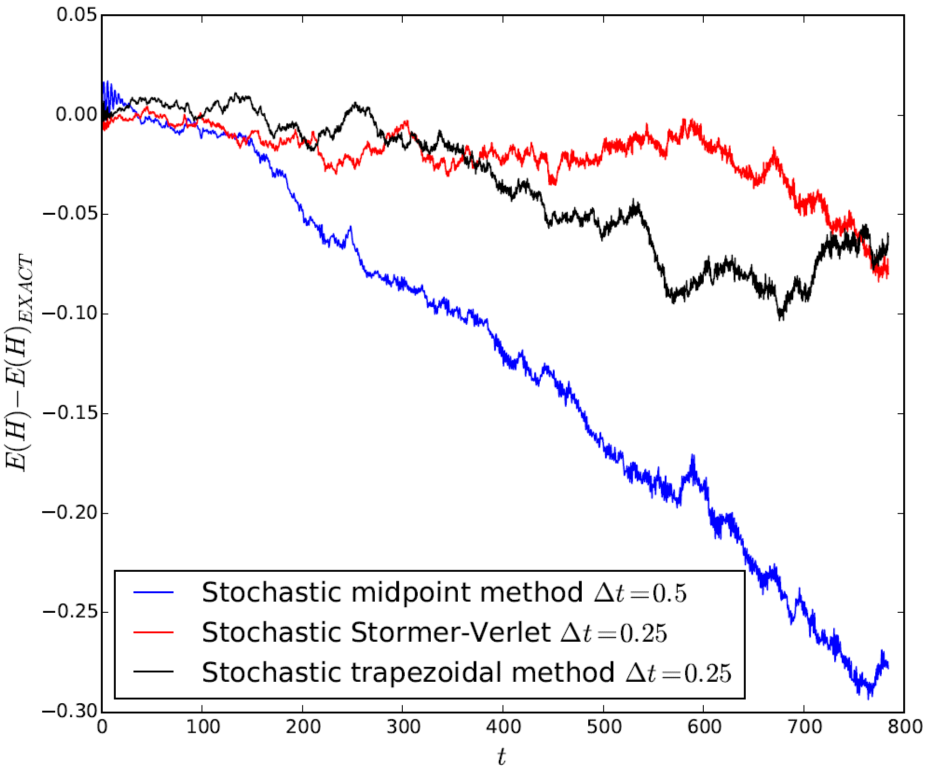
$$H(q,p) = \frac{1}{2}p^2 + \gamma q^4 \qquad h(q,p) = \beta q$$

- Expected value of the Hamiltonian

$$E(H) = H_0 + \frac{1}{2}\beta^2 t$$

Numerical tests: Hamiltonian behavior





Stochastic symplectic Runge-Kutta methods

- *s*-stage Runge-Kutta method *(Ma & Ding, 2015)*

$$Q_i = q_k + \Delta t \sum_{j=1}^{s} a_{ij} \frac{\partial H}{\partial p}(Q_j, P_j) + \Delta W \sum_{j=1}^{s} b_{ij} \frac{\partial h}{\partial p}(Q_j, P_j), \qquad i = 1, \ldots, s$$

$$P_i = p_k - \Delta t \sum_{j=1}^{s} \bar{a}_{ij} \frac{\partial H}{\partial q}(Q_j, P_j) - \Delta W \sum_{j=1}^{s} \bar{b}_{ij} \frac{\partial h}{\partial q}(Q_j, P_j), \qquad i = 1, \ldots, s$$

$$q_{k+1} = q_k + \Delta t \sum_{i=1}^{s} \alpha_i \frac{\partial H}{\partial p}(Q_i, P_i) + \Delta W \sum_{i=1}^{s} \beta_i \frac{\partial h}{\partial p}(Q_i, P_i)$$

$$p_{k+1} = p_k - \Delta t \sum_{i=1}^{s} \alpha_i \frac{\partial H}{\partial q}(Q_i, P_i) - \Delta W \sum_{i=1}^{s} \beta_i \frac{\partial h}{\partial q}(Q_i, P_i)$$

- Symplecticity conditions

$$\alpha_i \bar{a}_{ij} + \alpha_j a_{ji} = \alpha_i \alpha_j \qquad \beta_i \bar{a}_{ij} + \alpha_j b_{ji} = \beta_i \alpha_j$$

$$\alpha_i \bar{b}_{ij} + \beta_j a_{ji} = \alpha_i \beta_j \qquad \beta_i \bar{b}_{ij} + \beta_j b_{ji} = \beta_i \beta_j$$

Stochastic Galerkin methods as SPRK

Theorem 5. *Let* **r = s** *and let* $\bar{l}_{i,s-1}(\tau)$ *for* $i = 1, \ldots, s$ *denote the Lagrange polynomials of degree* $s-1$ *associated with the quadrature points* $0 \le c_1 < \ldots < c_s \le 1$*. Moreover, let the weights* α_i *be given by*

$$\alpha_i = \int_0^1 \bar{l}_{i,s-1}(\tau)\, d\tau,$$

and assume $\alpha_i \neq 0$ *for* $i = 1, \ldots, s$*. Then the stochastic Galerkin Hamiltonian variational integrator is equivalent to the stochastic partitioned Runge-Kutta method with the coefficients*

$$a_{ij} = \int_0^{c_i} \bar{l}_{j,s-1}(\tau)\, d\tau \qquad b_{ij} = \frac{\beta_j a_{ij}}{\alpha_j}$$

$$\bar{a}_{ij} = \frac{\alpha_j(\alpha_i - a_{ji})}{\alpha_i} \qquad \bar{b}_{ij} = \frac{\beta_j(\alpha_i - a_{ji})}{\alpha_i}$$

for $i, j = 1, \ldots, s$*.*

Methods of strong order 3/2

- Must include $\Delta Z = \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} dW(\xi)\, dt$
- For separable Hamiltonians $H(q,p) = T(p) + U(q)$

$$H_d^+(q_k, p_{k+1}) = \underset{\substack{q^1,\ldots,q^s \in Q \\ P_1,\ldots,P_r \in Q^* \\ q^0 = q_k}}{\text{ext}} \Big\{ p_{k+1} q^s - \Delta t \sum_{i=1}^{r} \Big[\bar{\alpha}_i P_i \dot{q}_d(t_k + c_i \Delta t) - \bar{\alpha}_i U\big(q_d(t_k + c_i \Delta t)\big) - \alpha_i T\big(P_i\big) \Big]$$

$$+ \Delta W \sum_{i=1}^{r} \bar{\beta}_i h\big(q_d(t_k + c_i \Delta t)\big) + \frac{\Delta Z}{\Delta t} \sum_{i=1}^{r} \bar{\gamma}_i h\big(q_d(t_k + c_i \Delta t)\big) \Big\}$$

where $\Delta W = \chi \sqrt{\Delta t}$ $\qquad \Delta Z = \frac{1}{2} \Delta t^{\frac{3}{2}} \Big(\chi + \frac{1}{\sqrt{3}} \eta \Big)$

- For $r = s$ one gets a type of Runge-Kutta methods
- *Milstein, Repin, Tretyakov, 2002* – method of order 3/2

THANK YOU!