Some aspects of representation theory of GL_n over non-archimedean field (joint with Alberto Mínguez)

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Let G be a reductive group over a non-archimedean field F.
Is there a "nice" class of irreducible representations of $G(F)$?
Knee-jerk reaction: Arthur's packets.
Problem: does not behave well with respect to Jacquet module.
As a trivial example, the Jacquet module of the trivial character of GL_2 with respect to the Borel subgroup is $|\cdot|^{-\frac{1}{2}} \otimes |\cdot|^{\frac{1}{2}}$ which is technically not of Arthur type.
More seriously, in GL_4, the representation

$$\mathrm{Sp}_{2,2} := \mathrm{socle}(\mathrm{Ind}_{P_{2,2}} |\det \cdot|^{\frac{1}{2}} \otimes |\det \cdot|^{-\frac{1}{2}})$$

is of Arthur type but

$$J_{P_{1,3}}(\mathrm{Sp}_{2,2}) = 1 \otimes \mathrm{socle}(\mathrm{Ind}_{P_{1,2}} |\cdot| \otimes |\det \cdot|^{-\frac{1}{2}})$$

which is not of Arthur type (even up to a twist).
For classical groups, the problem already arises for square-integrable representations.
Short answer to above question: I have no idea.

From now on, we only consider representations of $\mathrm{GL}_n(F)$ where F is a non-archimedean local field. (F is not necessarily commutative.)

- By a representation we will always mean a smooth complex representation of finite length.
- We write $\mathrm{Irr}_n = \mathrm{Irr}\,\mathrm{GL}_n(F)$ (equivalence classes of irreducible representations) and $\mathrm{Irr} = \cup_{n\geq 0}\mathrm{Irr}_n$.
- We denote by $\pi_1 \times \pi_2$ the representation parabolically induced from $\pi_1 \otimes \pi_2$ (normalized induction).
- If χ is a character of $Z(F)^*$ we write $\pi\chi$ for the twist of π by $\chi \circ \mathrm{Nrd}$.
- $\mathrm{soc}(\pi)$ is the socle of π, i.e., the sum of the irreducible subrepresentations of π (= the maximal semisimple subrepresentation of π). (If $\pi \neq 0$ then $\mathrm{soc}(\pi) \neq 0$.)

Lemma (Seok-Jin Kang, Masaki Kashiwara, Myungho Kim, Se-jin Oh (2014), in a slightly different context)
π_i representations of $\mathrm{GL}_{n_i}(F)$, $i = 1, 2, 3$.
$\sigma \subset \pi_1 \times \pi_2$
$\tau \subset \pi_2 \times \pi_3$
Assume $\sigma \times \pi_3 \subset \pi_1 \times \tau$ (submodules of $\pi_1 \times \pi_2 \times \pi_3$).
Then $\exists \omega \subset \pi_2$ such that $\sigma \subset \pi_1 \times \omega$ and $\omega \times \pi_3 \subset \tau$.

The proof is elementary.
More generally, (after discussions with Henniart)
Lemma
P, Q standard parabolic subgroups of G, $R = P \cap Q$
π a representation of M_R
$\sigma \subset I_R^P(\pi)$
$\tau \subset I_R^Q(\pi)$
Assume that $I_P^G(\sigma) \subset I_Q^G(\tau)$ (as subrepresentations of $I_R^G(\pi)$).
Then there exists $\kappa \subset \pi$ such that $\sigma \subset I_R^P(\kappa)$ and $I_R^Q(\kappa) \subset \tau$.

To any two representations π, σ of $\mathrm{GL}_{n_i}(F)$, $i = 1, 2$ we have standard intertwining operator

$$M_{\pi,\sigma}(s) : \pi \left|\cdot\right|^s \times \sigma \to \sigma \times \pi \left|\cdot\right|^s .$$

Let $r_{\pi,\sigma} = \mathrm{ord}_{s=0} M_{\pi,\sigma}(s) \geq 0$ and

$$R_{\pi,\sigma} = \lim_{s \to 0} s^{r_{\pi,\sigma}} M_{\pi,\sigma}(s) : \pi \times \sigma \to \sigma \times \pi$$

(defined and non-zero).
If $\tau \subset \pi$ then

$$M_{\pi,\sigma}(s)\big|_{\tau\left|\cdot\right|^s \times \sigma} = M_{\tau,\sigma}(s).$$

In particular, $r_{\tau,\sigma} \leq r_{\pi,\sigma}$ and

$$R_{\pi,\sigma}\big|_{\tau\times\sigma} = \begin{cases} R_{\tau,\sigma} : \tau \times \sigma \to \sigma \times \tau & r_{\tau,\sigma} = r_{\pi,\sigma}, \\ 0 & \text{otherwise.} \end{cases}$$

In any case $R_{\pi,\sigma}\big|_{\tau\times\sigma} : \tau \times \sigma \to \sigma \times \tau$. Also, we have

$$M_{\pi_1\times\pi_2,\pi_3}(s) = (M_{\pi_1,\pi_3}(s) \times \mathrm{id}_{\pi_2}) \circ (\mathrm{id}_{\pi_1} \times M_{\pi_2,\pi_3}(s)).$$

In particular, $r_{\pi_1\times\pi_2,\pi_3} \leq r_{\pi_1,\pi_3} + r_{\pi_2,\pi_3}$.

Theorem (KKKO) Let π be a (not necessarily irreducible) representation of $\mathrm{GL}_n(F)$ such that $R_{\pi,\pi}$ is a (non-zero) scalar. Then for any $\sigma \in \mathrm{Irr}$, $\mathrm{soc}(\pi \times \sigma)$ is irreducible and is equal to $\mathrm{Im}\, R_{\sigma,\pi}$. In particular, if moreover π is irreducible, so is $\pi \times \pi$.

Proof.

By assumption, $R_{\pi,\pi} = \lambda^{-1}\,\mathrm{id}_{\pi\times\pi}$ for some non-zero scalar λ. Therefore $r_{\pi\times\sigma,\pi} = r_{\pi,\pi} + r_{\sigma,\pi}$ and $\lambda R_{\pi\times\sigma,\pi} = \mathrm{id}_\pi \times R_{\sigma,\pi}$. Let $\tau \subset \pi \times \sigma$ irreducible. Then we have a commutative diagram

$$\begin{array}{ccc} \tau \times \pi & \xrightarrow{\lambda R_{\pi\times\sigma,\pi}\big|_{\tau\times\pi}} & \pi \times \tau \\ \downarrow & & \downarrow \\ \pi \times \sigma \times \pi & \xrightarrow{\mathrm{id}_\pi \times R_{\sigma,\pi}} & \pi \times \pi \times \sigma \end{array}$$

Hence, $\tau \times \pi \subset \pi \times R_{\sigma,\pi}^{-1}(\tau)$. By the previous lemma, $\exists \kappa \subset \sigma$ such that $\tau \subset \pi \times \kappa$ and $\kappa \times \pi \subset R_{\sigma,\pi}^{-1}(\tau)$. In particular, $\kappa \neq 0$ and therefore $\kappa = \sigma$. Thus $\mathrm{Im}\, R_{\sigma,\pi} \subset \tau$. Since τ was arbitrary, we conclude that $\mathrm{Im}\, R_{\sigma,\pi}$ is irreducible and is equal to $\mathrm{soc}(\pi \times \sigma)$. $\square$

They also show that under the above conditions $\mathrm{soc}(\pi \times \sigma)$ occurs with multiplicity one in $\mathrm{JH}(\pi \times \sigma)$. (Their argument carries over to this context, as explained to me by Max Gurevich.)
For simplicity, we say that π is SI if $\mathrm{soc}(\pi)$ is irreducible and occurs with multiplicity one in $\mathrm{JH}(\pi)$.
Definition We say that an irreducible π is LM if the following equivalent conditions are satisfied:

1. $\pi \times \pi$ is irreducible.
2. $\dim \mathrm{Hom}(\pi \times \pi, \pi \times \pi) = 1$.
3. $R_{\pi,\pi}$ is a scalar.
4. $\pi \times \sigma$ is SI for any $\sigma \in \mathrm{Irr}$.

Remark If F is commutative (and presumably in general), and $\pi_1, \pi_2 \in \mathrm{Irr}$ then any irreducible subrepresentation of $\pi_1 \times \pi_2$ is a quotient of $\pi_2 \times \pi_1$.
If π_1, π_2 are LM and $\pi_1 \times \pi_2$ is irreducible then $\pi_1 \times \pi_2$ is LM.
If π is LM and $\sigma \in \mathrm{Irr}$ then
$\pi \times \sigma \in \mathrm{Irr} \iff \mathrm{soc}(\pi \times \sigma) = \mathrm{soc}(\sigma \times \pi)$.

Classification of Irr (Bernstein–Zelevinsky, ...)

Let $\rho \in \mathrm{Irr}$ be supercuspidal.

- $\rho \times \rho$ is irreducible.
  This is done by explicating $M_{\rho|\cdot|^s \times \rho}$ and showing that its residue at $s = 0$ is a non-zero scalar (Olshanski 1974, proof streamlined by Shahidi 2000). The proof is elementary – it relies on the simple nature of the action of $\mathrm{GL}_m \times \mathrm{GL}_m$ on Mat_m by left and right multiplication.
- $\exists! s_\rho \in \mathbb{R}_{>0}$ such that $\rho\,|\cdot|^{s_\rho} \times \rho$ is reducible.
  We write $\nu_\rho = |\cdot|^{s_\rho}$, $\overrightarrow{\rho} = \rho\nu_\rho$, $\overleftarrow{\rho} = \rho\nu_\rho^{-1}$.
  Note that $\nu_{\rho\chi} = \nu_\rho$ for any χ and $\nu_{\rho^\vee} = \nu_\rho$. (If F is commutative, $s_\rho = 1$. In general $s_\rho \in \mathbb{Z}_{>0}$ but this fact requires more input and does not play a role in the classification.)
- If $\rho_1, \rho_2 \in \mathrm{Irr}$ are supercuspidal then $\rho_1 \times \rho_2$ is reducible if and only if $\rho_2 = \rho_1\nu_{\rho_1}^{\pm 1}$.
  (This uses Casselman's unitarity argument.)

- I'm not aware of a purely algebraic proof of the last two parts, unless one takes for granted the classification of supercuspidal representations due to Bushnell–Kutzko (1993; F commutative), Sécherre, Sécherre–Stevens, (2004–2012; F non-commutative).
- A segment is a non-empty set of the form $\Delta = \{\rho_1, \dots, \rho_k\}$ where $\rho_i \in \mathrm{Irr}$ is supercuspidal and $\rho_{i+1} = \overrightarrow{\rho}_i$. In this case $\rho_1 \times \cdots \times \rho_k$ is SI. Let $Z(\Delta) = \mathrm{soc}(\rho_1 \times \cdots \times \rho_k)$.
- We have $Z(\Delta)^\vee = Z(\Delta^\vee)$ where $\Delta^\vee = \{\rho_k^\vee, \dots, \rho_1^\vee\}$.
- Notation: $b(\Delta) = \rho_1$, $e(\Delta) = \rho_k$, $\overleftarrow{\Delta} = \{\overleftarrow{\rho_1}, \dots, \overleftarrow{\rho_k}\}$, $\overrightarrow{\Delta} = \{\overrightarrow{\rho_1}, \dots, \overrightarrow{\rho_k}\}$, ${}^-\Delta = \{\rho_2, \dots, \rho_k\}$, $\Delta^- = \{\rho_1, \dots, \rho_{k-1}\}$.
- Δ_1 and Δ_2 are linked if $\Delta_1 \cup \Delta_2$ is a segment which properly contains each of Δ_1 and Δ_2. In this case, either $b(\Delta_2) \in \overrightarrow{\Delta_1}$ (and we write $\Delta_1 \prec \Delta_2$) or the symmetric condition.

The classification theorem

A multisegment is a formal sum $\mathfrak{m} = \Delta_1 + \cdots + \Delta_N$ of segments. Assume that $\Delta_1, \ldots, \Delta_N$ is a sequence of segments such that $\Delta_i \not\prec \Delta_j$ for all $i < j$. Then

$$\zeta(\mathfrak{m}) := Z(\Delta_1) \times \cdots \times Z(\Delta_N) \quad \text{(standard module)}$$

is SI and depends only on $\mathfrak{m} = \Delta_1 + \cdots + \Delta_N$. The map

$$\boxed{\mathfrak{m} \mapsto Z(\mathfrak{m}) := \mathrm{soc}(\zeta(\mathfrak{m}))}$$

defines a bijection between multisegments and Irr.
(F commutative – Bernstein–Zelevinsky (1977), Zelevinsky (1980).
General case: Deligne–Kazhdan–Vigneras (1984), Tadić (1990), Badulescu (2002), Mínguez–Sécherre (2013), ... For a uniform approach with minimal prerequisites: Appendix of 1411.6310 (•–Mínguez–Tadić))

Some properties of $Z(\mathfrak{m})$

- $Z(\mathfrak{m}) = \zeta(\mathfrak{m}) \iff$ the Δ_i's in $\mathfrak{m}$ are mutually unlinked.
- The elements of $\mathrm{JH}(\zeta(\mathfrak{m}))$ are of the form $Z(\mathfrak{n})$ where $\mathfrak{n}$ is obtained from $\mathfrak{m}$ by "union-intersection". The multiplicities are more subtle. (More about that later.)
- $Z(\mathfrak{m})^\vee = Z(\mathfrak{m}^\vee)$ where $(\Delta_1 + \cdots + \Delta_N)^\vee := \Delta_1^\vee + \cdots + \Delta_N^\vee$.
- If we invert ν_ρ we get the Langlands classification $\mathfrak{m} \mapsto \mathrm{L}(\mathfrak{m})$.
- $Z(\mathfrak{m} + \mathfrak{n})$ occurs with multiplicity one in $\mathrm{JH}(Z(\mathfrak{m}) \times Z(\mathfrak{n}))$.
- Main question left open: what is the decomposition of $Z(\mathfrak{m}) \times Z(\mathfrak{n})$? In particular, when is it irreducible?
  (Very) recent thesis by Deng Taiwang may shed light on this.
  Note that $Z(\mathfrak{m}) \times Z(\mathfrak{n})$ is irreducible if and only if

  $$(Z(\mathfrak{socz}(\mathfrak{m}, \mathfrak{n})) :=) \, \mathrm{soc}(Z(\mathfrak{m}) \times Z(\mathfrak{n})) = Z(\mathfrak{m} + \mathfrak{n}) \qquad \text{(LI)}$$

  (if $Z(\mathfrak{m})$ is LM) and $Z(\mathfrak{m} + \mathfrak{n}) = \mathrm{soc}(Z(\mathfrak{n}) \times Z(\mathfrak{m}))$.
  A sufficient, but by no means necessary condition for (LI) is that $\Delta \not\prec \Delta'$ for any $\Delta \in \mathfrak{m}$, $\Delta' \in \mathfrak{n}$.

From now on we fix $\rho \in \mathrm{Irr}$ supercuspidal and only consider segments and multisegments supported in the cuspidal line of ρ

$$\mathbb{Z}_\rho = \{\rho\nu_\rho^n : n \in \mathbb{Z}\}.$$

For integers $a \le b$ we write $[a, b] = [a, b]_\rho = \{\rho\nu_\rho^a, \dots, \rho\nu_\rho^b\}$.
Since ρ is fixed throughout, we will suppress it from the notation.
(For all practical purposes ρ is the trivial character of GL_1.)
In fact ρ is immaterial – the Hecke algebra pertaining to the Bernstein component of $\overbrace{\rho \otimes \cdots \otimes \rho}^{m}$ is essentially independent of ρ (i.e., it is the usual Iwahori–Hecke algebra for S_m) (either type theory or Heiermann, 2011).
It will be convenient to set $Z([a, a-1]) = 1$ (the irreducible one-dimensional representation of GL_0) and $Z([a, b]) = 0$ (the trivial, zero-dimensional representation) if $b < a - 1$.
We also write $\Delta_1 \le_b \Delta_2$ if either $b(\Delta_1) < b(\Delta_2)$ or $(b(\Delta_1) = b(\Delta_2)$ and $e(\Delta_1) \le e(\Delta_2))$. Similarly, for $\Delta_1 \le_e \Delta_2$.

There are examples of irreducible π's which are not LM. The first one was constructed by Leclerc (2003), giving a counterexample to a conjecture by A. Berenstein–Zelevinsky (1993). It is given by $Z(\mathfrak{m})$ where

$$\mathfrak{m} = [1,2] + [-1,1] + [0,0] + [-2,-1].$$

We remark that

$$\mathrm{JH}(\mathrm{Sp}_{2,2} \left|\det \cdot\right| \times \mathrm{Sp}_{2,2} \left|\det \cdot\right|^{-1})$$

$$\text{i.e., } \mathrm{JH}\left(Z([-1,0]+[-2,-1]) \times Z([1,2]+[0,1])\right)$$

$$= Z(\mathfrak{m}) + Z([1,2]+[0,1]+[-1,0]+[-2,-1]) + Z([-1,2]+[-2,1]).$$

and $Z(\mathfrak{m}) \times Z(\mathfrak{m})$ is semisimple of length two. It decomposes as

$$Z(\mathfrak{m}+\mathfrak{m}) \oplus Z([1,2]+[0,1]+[-1,0]+[-2,-1]) \times Z([-1,2]+[-2,1]).$$

I do not know whether $\pi \times \pi$ is semisimple for all $\pi \in \mathrm{Irr}$.

Ladder representations

Ladder multisegments are those of the form

$$\Delta_1 + \cdots + \Delta_N$$

$\Delta_i = [a_i, b_i]$ where $a_1 > \cdots > a_N$ and $b_1 > \cdots > b_N$.
A particularly important example: $a_{i+1} = a_i - 1$, $b_{i+1} = b_i - 1$.
(Speh representations)

- A ladder representation is LM.
- Determinantal formula (in the Grothendieck group):

$$Z(\mathfrak{m}) = \sum_{w \in S_N} \operatorname{sgn} w \; Z([a_1, b_{w(1)}]) \times \cdots \times Z([a_N, b_{w(N)}])$$
$$= \det(Z([a_i, b_j])).$$

- Any Jacquet module of a ladder representation is a direct sum of ladder representations (explicitly described) with distinct cuspidal support.

Example

The Jacquet module is the tensor product of the four ladders composed of the four different colors.

Roughly speaking, ladder representations behave very much like square-integrable (or segment) representations.

Theorem (Max Gurevich 2016) The product of two ladder representations is multiplicity free.

Mœglin–Waldspurger algorithm (1986)

Let $\mathfrak{m} = \Delta_1 + \cdots + \Delta_N$ with $\Delta_1 \geq_e \cdots \geq_e \Delta_N$. Let $j_0 = 1$ and

$$\text{inductively:} \quad j_l = \min\{j : \Delta_j \prec \Delta_{j_{l-1}} \text{ and } e(\Delta_j) = e(\overleftarrow{\Delta}_{j_{l-1}})\}.$$

Let $k \geq 0$ be the last index for which j_l is defined. Set

$$\mathfrak{m}^- = \mathfrak{m} - \sum_{l=0}^{k} \Delta_{j_l} + \sum_{l=0}^{k} \Delta_{j_l}^- \quad \text{and} \quad \Delta(\mathfrak{m}) = [e(\Delta_{j_k}), e(\Delta_{j_0})].$$

We define recursively $\boxed{\mathfrak{m}^t = \Delta(\mathfrak{m}) + (\mathfrak{m}^-)^t.}$
We have $Z(\mathfrak{m}^t) = \mathrm{L}(\mathfrak{m})$, $\mathrm{L}(\mathfrak{m}^t) = Z(\mathfrak{m})$.
The set of ladders is preserved under $\mathfrak{m} \mapsto \mathfrak{m}^t$.
The inverse operation will be denote by $\mathfrak{n}_\Delta^+$. Thus, $\mathfrak{m} = (\mathfrak{m}^-)^+_{\Delta(\mathfrak{m})}$.

Combinatorial description of $\mathrm{soc}(\pi \times \sigma)$, π ladder

Suppose that $\mathfrak{m} = \Delta_1 + \cdots + \Delta_N$ is a ladder with $\Delta_1 >_e \cdots >_e \Delta_N$ and $\mathfrak{n} = \Delta'_1 + \cdots + \Delta'_{N'}$ is any multisegment with $\Delta'_1 \geq_e \cdots \geq_e \Delta'_{N'}$. We can describe $\mathfrak{socz}(\mathfrak{m}, \mathfrak{n})$ inductively as follows (1411.6310). If $e(\Delta'_1) \leq e(\Delta_1)$ and k is the last index (possibly 0) such that $\Delta'_k \geq_e \Delta_1$ then upon writing

$$\mathfrak{m} = \Delta_1 + \mathfrak{m}', \quad \mathfrak{n} = \Delta'_1 + \cdots + \Delta'_k + \mathfrak{n}'$$

we have

$$\mathfrak{socz}(\mathfrak{m}, \mathfrak{n}) = \Delta'_1 + \cdots + \Delta'_k + \Delta_1 + \mathfrak{socz}(\mathfrak{m}', \mathfrak{n}').$$

Otherwise (i.e., if $e(\Delta'_1) > e(\Delta_1)$)

$$\mathfrak{socz}(\mathfrak{m}, \mathfrak{n}) = \mathfrak{socz}(\mathfrak{m}, \mathfrak{n}^-)^+_{\Delta(\mathfrak{n})}.$$

Recall that by definition

$$Z(\mathfrak{socz}(\mathfrak{m}, \mathfrak{n})) = \mathrm{soc}(Z(\mathfrak{m}) \times Z(\mathfrak{n})).$$

Irreducibility criterion for $Z(\mathfrak{m}) \times Z(\mathfrak{n})$, $\mathfrak{m}$ ladder

As before, $\mathfrak{m} = \Delta_1 + \cdots + \Delta_N$ is a ladder with $\Delta_1 >_e \cdots >_e \Delta_N$ and $\mathfrak{n} = \Delta'_1 + \cdots + \Delta'_{N'}$ is any multisegment with $\Delta'_1 \geq_e \cdots \geq_e \Delta'_{N'}$. Recall

$$Z(\mathfrak{m}) \times Z(\mathfrak{n}) \text{ is irreducible } \iff \mathfrak{soc}_{\mathfrak{z}}(\mathfrak{m}, \mathfrak{n}) = \mathfrak{m} + \mathfrak{n} = \mathfrak{soc}_{\mathfrak{z}}(\mathfrak{n}, \mathfrak{m}).$$

We can explicate the condition $\mathfrak{soc}_{\mathfrak{z}}(\mathfrak{m}, \mathfrak{n}) = \mathfrak{m} + \mathfrak{n}$ as follows. Let

$$X = \{(i,j) \in [N] \times [N'] : \Delta_i \prec \Delta'_j\}, \; Y = \{(i,j) \in [N] \times [N'] : \overleftarrow{\Delta}_i \prec \Delta'_j\}$$

($[N] = \{1, \ldots, N\}$). Define a relation $\rightsquigarrow$ between Y and X by

$$(i_2, j_2) \rightsquigarrow (i_1, j_1) \iff \text{either} \begin{cases} i_1 = i_2 \text{ and } \Delta'_{j_2} \prec \Delta'_{j_1}, \text{ or} \\ j_1 = j_2 \text{ and } \Delta_{i_1} \prec \Delta_{i_2} \end{cases}$$

Then $\mathfrak{soc}_{\mathfrak{z}}(\mathfrak{m}, \mathfrak{n}) = \mathfrak{m} + \mathfrak{n}$ if and only if there exists an injective function $f : X \to Y$ such that $f(i,j) \rightsquigarrow (i,j)$ for all $(i,j) \in X$.

Unitary dual of $GL_n(F)$

(Bernstein (1984); Tadić (1986, 1990);
Badulescu–Henniart–Lemaire–Sécherre (2010))

Once again, it is possible to give a uniform treatment of the unitary dual with minimal sine qua nons 1411.6310.

One of the key facts is that a Speh representation $\times$ unitarizable representation is irreducible. This follows immediately from the fact that a Speh representation is LM and unitarizable.

Relation to category $\mathcal{O}$ (Arakawa–Suzuki (1998))

(See also Henderson (2007), Barbasch–Ciubotaru (2015))
Let $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ with $\lambda_1 \geq \cdots \geq \lambda_N$. There is an exact functor F_λ from category $\mathcal{O}$ with respect to $\mathfrak{gl}_N$ to representations supported in the cuspidal line of ρ. It satisfies $F_\lambda(M(\mu)) = \zeta(\mathfrak{m}_{\mu,\lambda})$

$$\text{and} \quad F_\lambda(L(\mu)) = \begin{cases} Z(\mathfrak{m}_{\mu,\lambda}) & \text{if } \mu_i \leq \mu_{i+1} \text{ whenever } \lambda_i = \lambda_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{m}_{\mu,\lambda} = \sum_{i=1}^{N} [\mu_i, \lambda_i]$. We may apply the Kazhdan–Lusztig conjecture (proved by Beilinson–Bernstein, Brylinski–Kashiwara (1981)) expressing the change of basis matrix between the irreducible modules $L(\mu)$ and the Verma modules $M(\mu)$ in category $\mathcal{O}$ in terms of the Kazhdan–Lusztig polynomials. Then we can apply F_λ.
Let $\mu = (\mu_1, \ldots, \mu_N) \in \mathbb{Z}^N$ with $\mu_1 \geq \cdots \geq \mu_N$. Note that $\mathfrak{m}_{w\mu,\lambda} \leq \mathfrak{m}_{w'\mu,\lambda}$ in the Zelevinksy order (i.e., $Z(\mathfrak{m}_{w'\mu,\lambda})$ occurs in $\zeta(\mathfrak{m}_{w\mu,\lambda})$) if and only if $w \leq w'$ in the Bruhat order of S_N.

Let S_λ = stabilizer of λ in S_N. We get that for any $\mu = (\mu_1, \dots, \mu_N) \in \mathbb{Z}^N$ with $\mu_1 \geq \cdots \geq \mu_N$ and $w \in S_N$ we have

$$\boxed{\zeta(\mathfrak{m}_{w\mu,\lambda}) = \sum_{w' \in [S_\lambda \backslash S_N / S_\mu]} P_{w,w'}(1) Z(\mathfrak{m}_{w'\mu,\lambda})}$$

where $[S_\lambda \backslash S_N / S_\mu] = \{w' \in S_N \text{ of maximal length in } S_\lambda w' S_\mu\}$

and $P_{w,w'}$ are the Kazhdan–Lusztig polynomials wrt S_N.
Equivalently, for $w \in [S_\lambda \backslash S_N / S_\mu]$ we have

$$\boxed{Z(\mathfrak{m}_{w\mu,\lambda}) = \sum_{w' \in S_N} \operatorname{sgn} w'w \; P_{w'w_0, ww_0}(1) \; \zeta(\mathfrak{m}_{w'\mu,\lambda})}$$

where w_0 = longest element in S_N.
Ladder representations correspond to $S_\lambda = S_\mu = 1$ and $w = 1$.
The determinantal formula reflects the fact that $P_{w',w_0} = 1$ for all $w' \in S_N$. In fact, one gets a resolution of a ladder by standard modules (corresponding to the BGG resolution).

The fact that a ladder is LM is translated to the following identities. Let H be the parabolic subgroup of S_{2N}

$$H = \{w \in S_{2N} : \{w(2i-1), w(2i)\} = \{2i-1, 2i\}\ \forall i\} \simeq S_2^N.$$

$$H\backslash S_{2N}/H \leftrightarrow \{M \in \mathrm{Mat}_N(\mathbb{Z}_{\geq 0}) : \sum_{i=1}^{N} M_{i,j} = \sum_{l=1}^{N} M_{k,l} = 2\ \forall j, k\}$$

$= \{\, T_{\sigma_1} + T_{\sigma_2} : \sigma_1, \sigma_2 \in S_N\}$ where $T_\sigma =$ permutation matrix of σ.

$w \mapsto \sigma_1\sigma_2^{-1}$ defines a map $H\backslash S_{2N}/H \to conj.cls.(S_N)$, $w \mapsto [w]$.

$$\text{Then} \quad \boxed{\sum_{x \in HwH} \operatorname{sgn} x\ P_{x,\widetilde{w_0}}(1) = \operatorname{sgn}[w]\ 2^r \quad \forall w \in H\backslash S_{2N}/H}$$

where r is the number of cycles of size > 1 in $[w]$ and for any $\sigma \in S_N$, $\tilde{\sigma} \in S_{2N}$ is the "doubled" permutation $\tilde{\sigma}(2i-1) = 2\sigma(i) - 1$, $\tilde{\sigma}(2i) = 2\sigma(i)$.

Remark The map $w \mapsto [w]$ factors through a bijection $K \backslash S_{2N}/K \to conj.cls.(S_N)$ where

$$K = N_{S_{2N}}(H) = H \rtimes \widetilde{S_N} = C_{S_{2N}}(2i-1 \mapsto 2i, 2i \mapsto 2i-1).$$

I am not aware of an independent proof of the relation

$$\sum_{x \in HwH} \operatorname{sgn} x \; P_{x,\widetilde{w_0}}(1) = \operatorname{sgn}[w] \; 2^r$$

even for $w = \mathrm{id}$ or for the fact that the left-hand side depends only on KwK.

Note that $[w] = 1$ if and only if $w \in K$. In this case it seems (empirically) that

$$\sum_{x \in H\tilde{\sigma}} \operatorname{sgn} x \; P_{x,\widetilde{w_0}}(q) = q^{\ell(w_0\sigma)} \quad \forall \sigma \in S_N.$$

I do not have a good guess for $\sum_{x \in HwH} \operatorname{sgn} x \; P_{x,\widetilde{w_0}}(q)$ in general.

In general, one can in principle detect the irreducibility of $Z(\mathfrak{m}) \times Z(\mathfrak{n})$ by comparing the expansion of $Z(\mathfrak{m} + \mathfrak{n})$ in terms of the standard modules to that of $Z(\mathfrak{m}) \times Z(\mathfrak{n})$. Needless to say that this is not very practical for many segments. However, it is feasible when the total number of segments is, say, ≤ 12. (I used code by Greg Warrington (2010; implementing, I believe, an algorithm of Brenti (1998?)) for KL polynomials of type A.)

Let us consider multisegments $\mathfrak{m} = \sum_{i=1}^{N} [a_i, b_i]$ such that the a_i's are distinct and the b_i's are distinct. (We call them regular.)
We can start with $a_1 < \cdots < a_N$, $b_1 > \cdots > b_N$ and consider

$$\mathfrak{m}_\sigma = \sum_{i=1}^{N} [a_{\sigma^{-1}(i)}, b_i], \quad \sigma \in S_N.$$

Sometimes $\mathfrak{m}_\sigma$ will be void (i.e., if $b_i < a_{\sigma^{-1}(i)} - 1$ for some i). If $\mathfrak{m}_{w_0}$ is void (i.e., if $b_i < a_{N+1-i} - 1$ for some i) then all $\mathfrak{m}_\sigma$'s are void. We exclude this case. Otherwise, there exists a unique $\sigma_0 \in S_N$ such that $\mathfrak{m}_{\sigma_0}$ is not void, and consists of pairwise unlinked segments. (If $a_N \leq b_N + 1$ then $\sigma_0 = \mathrm{id}$. This is the "saturated" case.) Then $\mathfrak{m}_\sigma$ is not void if and only if $\sigma \geq \sigma_0$ in the Bruhat order.
Example: $N = 4$, $a_i = -b_i = i - 3$. Then $\sigma_0 = (1243)$,
$\mathfrak{m}_{\sigma_0} = [-2, 2] + [-1, 1] + \cancel{[1, 0]} + \cancel{[0, -1]}$.

For any $\sigma \in S_N$ let X_σ be the Schubert variety corresponding to σ, i.e., the closure of the Schubert cell $Y_\sigma = B_N(\mathbb{C})\backslash B_N(\mathbb{C})T_\sigma B_N(\mathbb{C})$ in the flag variety $B_N(\mathbb{C})\backslash \mathrm{GL}_N(\mathbb{C})$.
Thus, X_{id} is a point and $X_{w_0} = B_N(\mathbb{C})\backslash \mathrm{GL}_N(\mathbb{C})$.
In general X_σ is projective, but not smooth.
$X_\tau \subset X_\sigma \iff \tau \le \sigma$ in the Bruhat order.
The following conditions are equivalent for any $\sigma_0 \le \sigma$.

1. Y_{σ_0} is contained in the smooth locus of X_σ.
2. $P_{\sigma_0,\sigma} = 1$.
3. $P_{\sigma',\sigma} = 1$ for all $\sigma_0 \le \sigma' \le \sigma$.
4. $\#\{i < j : \sigma_0(i) > \sigma_0(j) \text{ and } \sigma_0\tau_{i,j} \le \sigma\} = \ell(\sigma) - \ell(\sigma_0)$ where $\tau_{i,j}$ is the transposition $i \leftrightarrow j$.

(Lakshmibai–Seshadri (1984), Deodhar (1985)) For the σ_0 above the conditions are further equivalent to the "determinantal" formula

$$Z(\mathfrak{m}_\sigma) = \sum_{\sigma_0 \le \sigma' \le \sigma} \operatorname{sgn} \sigma\sigma' \; \zeta(\mathfrak{m}_{\sigma'}).$$

Example

As before, consider the case $N = 4$, $a_i = -b_i = i - 3$, $\sigma_0 = (1243)$. Let $\sigma = (4231)$. Then

$$\mathfrak{m}_\sigma = [1, 2] + [-1, 1] + [0, 0] + [-2, -1].$$

We have $P_{\sigma_0,\sigma}(q) = 1 + q$. Note that the above is Leclerc's example. Is it a coincidence?

Theorem Let $a_1 < \cdots < a_N$, $b_1 > \cdots > b_N$ with $a_{N+1-i} \leq b_i + 1$ for all i. Let $\sigma_0 \in S_N$ be as above (i.e., $\mathfrak{m}_\sigma = \sum_{i=1}^{N} [a_{\sigma^{-1}(i)}, b_i]$ is not void if and only if $\sigma \geq \sigma_0$).
Assume that $\sigma \geq \sigma_0$ and X_σ is smooth at Y_{σ_0}. Then $Z(\mathfrak{m}_\sigma)$ is LM (i.e. $Z(\mathfrak{m}_\sigma) \times Z(\mathfrak{m}_\sigma)$ is irreducible). Equivalently,

$$\sum_{x \in HwH} \operatorname{sgn} x \ P_{x,\tilde{\sigma}}(1) = \operatorname{sgn}[w] \ 2^r$$

for all $w \in S_{2N}$ such that $w \geq \widetilde{\sigma_0}$.
Remark We expect that the converse also holds. (I checked it completely up to $N \leq 5$ and partially for $N = 6$.)

Lemma Let $\pi \in \mathrm{Irr}$. Assume $\exists \pi_1, \pi_2 \in \mathrm{Irr}$ such that:

- $\pi \hookrightarrow \pi_1 \times \pi_2$,
- π_i is LM, $i = 1, 2$,
- $\pi \times \pi_1$ is irreducible.

Then π is LM.

Proof

$$\pi \times \pi \hookrightarrow \pi \times \pi_1 \times \pi_2 \simeq \pi_1 \times \pi \times \pi_2 \hookrightarrow \pi_1 \times \pi_1 \times \pi_2 \times \pi_2.$$

By assumption, $\pi_1 \times \pi_1$ is LM and $\pi_2 \times \pi_2$ is irreducible. Hence $\pi_1 \times \pi_1 \times \pi_2 \times \pi_2$ is SI, and a fortiori $\pi \times \pi$. $\square$

The theorem is proved by induction. Using the lemma above, and the combinatorial criterion for the irreducibility of ladder$\times\pi$, the induction step reduces to a combinatorial statement. (We take π_1 to be a suitable ladder.)

Odds and ends

Let $\tau = (\tau_1 \dots \tau_l) \in S_l$ and $\sigma = (\sigma_1 \dots \sigma_n) \in S_n$, $n \geq l$. We say that τ *occurs* in σ if there exist indices $i_1 < \cdots < i_l$ such that the order type of $\sigma_{i_1}, \dots, \sigma_{i_l}$ coincides with τ, that is $\sigma_{i_j} < \sigma_{i_k}$ if and only if $\tau_j < \tau_k$. In other words, the permutation matrix of τ occurs as a $l \times l$-minor in the permutation matrix of σ. For instance,

$$\tau = (4231) \text{ occurs in } \sigma = (\underline{7}4\underline{3}2\underline{6}8\underline{1}5).$$

If τ does not occur in σ, we say that σ avoids τ.
Going back to the previous setup, assume now that $\sigma_0 = \mathrm{id}$ (i.e., that $a_N \leq b_N + 1$). Then X_σ is smooth at X_{σ_0} if and only if X_σ is smooth (if and only if $P_{e,\sigma} = 1$). This happens if and only if σ avoids (4231) and (3412) Lakshmibai–Sandhya (1990). Suppose that this is the case. By the theorem

$$\sum_{x \in HwH} \operatorname{sgn} x \; P_{x,\tilde{\sigma}}(1) = \operatorname{sgn}[w] \, 2^r.$$

Suppose that $\sigma \in S_N$ is smooth. Thus,

$$\sum_{x \in HwH} \operatorname{sgn} x \ P_{x,\tilde{\sigma}}(1) = \operatorname{sgn}[w] \ 2^r. \tag{1}$$

Can one see it directly?
The permutation $\tilde{\sigma}$ (if $\sigma \neq \mathrm{id}$) is never (3412) avoiding, and therefore Lascoux's formula for the KL polynomials (1995, following Lascoux–Schützenberger 1981) is unfortunately not applicable. However, following Deodhar (1990), it is also easy to compute $P_{w',w}$ if the Bott–Samelson resolution of X_w is small. This condition was characterized combinatorially by Billey–Warrington (2001). It holds if and only if w avoids the patterns (321), (46718235), (46781235), (56718234), (56781234). In particular, for $\tilde{\sigma}$, this is the case if and only if σ avoids (321) and (3412). These are precisely the products of distinct simple reflections. They are all smooth, and their number is F_{2N-1} where F_n is the n'th Fibonacci number (Fan 1998). It is a not difficult to check combinatorially that (1) holds for these σ's.

What about the non-regular case?
The "obvious" guess (in the saturated case) would be that $Z(\mathfrak{m}_\sigma)$ is LM if and only if the closure of $P_{\{a_1,\dots,a_N\}}\sigma P_{\{b_1,\dots,b_N\}}$ is smooth. ($P_{\{a_1,\dots,a_N\}}$ is the standard parabolic subgroups of $\mathrm{GL}_N(\mathbb{C})$ whose simple roots correspond to the i's such that $a_i = a_{i+1}$.)
Unfortunately, there are counterexamples in both directions for $N = 5$. At the moment we do not have a reasonable conjecture.

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It's a girl!