On the local Langlands correspondence and Arthur conjecture for orthogonal groups

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Contents

Notation

- F non-arch. local field, $\mathrm{char}(F) = 0$;
- $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ Weil–Deligne group of F;
- G connected reductive quasi-split group over F;
- $\widehat{G}$ the complex dual group of G;
- ${}^L G = \widehat{G} \rtimes W_F$ the L-group of G.

L-parameters and component groups

We say that a homomorphism $\varphi\colon WD_F \to {}^L G$ is admissible if

$$\begin{array}{ccc} WD_F = W_F \times \mathrm{SL}_2(\mathbb{C}) & \xrightarrow{\quad\varphi\quad} & {}^L G = \widehat{G} \rtimes W_F \\ & \searrow \quad \swarrow & \\ & W_F & \end{array}$$

is commutative.
The set of L-parameters of G is defined by

$$\Phi(G) = \{\varphi\colon WD_F \to {}^L G \text{ adm. hom.}\}/(\widehat{G}\text{-conjugacy}).$$

The component group of $\varphi \in \Phi(G)$ is defined by

$$\mathcal{S}_\varphi = \pi_0(\mathrm{Cent}(\mathrm{Im}(\varphi), \widehat{G})/Z(\widehat{G})^{W_F}).$$

Local Langlands conjecture (LLC)

Langlands predicted a classification of

$$\mathrm{Irr}(G(F)) = \{\text{irreducible rep. of } G(F)\}$$

in terms of

$$\Phi(G) \quad \text{and} \quad \mathrm{Irr}(\mathcal{S}_\varphi).$$

LLC for G classical groups is established by Arthur, Mœgline, Mok, and Kaletha–Minguez–Shin–White.

Example: $G = \mathrm{SO}(2n+1)$

When $G = \mathrm{SO}(2n+1)$ split,

- ${}^L G = \mathrm{Sp}(2n, \mathbb{C}) \times W_F$;
- $\Phi(\mathrm{SO}(2n+1)) \xrightarrow{1:1} \{\phi\colon WD_F \to \mathrm{Sp}(2n, \mathbb{C})\}$;
- $A_\phi = \pi_0(\mathrm{Cent}(\mathrm{Im}(\phi), \mathrm{Sp}(2n, \mathbb{C}))/\{\pm 1\}) = \mathcal{S}_\varphi (\cong (\mathbb{Z}/2\mathbb{Z})^r)$.

LLC for $\mathrm{SO}(2n+1)$ (Arthur)

1. There exists a surjection

$$\mathrm{Irr}(\mathrm{SO}(2n+1, F)) \twoheadrightarrow \Phi(\mathrm{SO}(2n+1)).$$

   The fiber of ϕ is denoted by Π_ϕ^0 and called the L-packet of ϕ.

2. There exists a bijection

$$\iota\colon \Pi_\phi^0 \xrightarrow{1:1} \widehat{A_\phi},\ \sigma^0 \mapsto \iota(\sigma^0).$$

Example: $G = \mathrm{SO}(2n)$

When $G = \mathrm{SO}(2n)$ quasi-split over F and split over E/F,

- ${}^L G = \mathrm{SO}(2n, \mathbb{C}) \rtimes W_F$.
- Put

$$\Phi(\mathrm{SO}(2n))/\sim := \{\phi \colon WD_F \to \mathrm{O}(2n, \mathbb{C}) \mid \det(\phi) = \omega_{E/F}\}.$$

  Then $\Phi(\mathrm{SO}(2n)) \twoheadrightarrow \Phi(\mathrm{SO}(2n))/\sim$.

- $A_\phi = \pi_0(\mathrm{Cent}(\mathrm{Im}(\phi), \mathrm{O}(2n, \mathbb{C}))/\{\pm 1\})(\cong (\mathbb{Z}/2\mathbb{Z})^r)$.
- $A_\phi^+ = \pi_0(\mathrm{Cent}(\mathrm{Im}(\phi), \mathrm{SO}(2n, \mathbb{C}))/\{\pm 1\}) = \mathcal{S}_\varphi$.

Then

$$1 \longrightarrow A_\phi^+ \longrightarrow A_\phi \xrightarrow{\det} \mathbb{Z}/2\mathbb{Z} = \{0, 1\}.$$

Weak LLC for $\mathrm{SO}(2n)$

Since $\Phi(\mathrm{SO}(2n)) \twoheadrightarrow \Phi(\mathrm{SO}(2n))/\sim$ is not injective,
Arthur established a weaker version of LLC for $\mathrm{SO}(2n)$.

Weak LLC for $\mathrm{SO}(2n)$ (Arthur)

1. There exists a surjection

$$\mathrm{Irr}(\mathrm{SO}(2n,F))/\mathrm{O}(2n,F) \twoheadrightarrow \Phi(\mathrm{SO}(2n))/\sim .$$

   The fiber of ϕ is denoted by Π_ϕ^0 and called the L-packet of ϕ.

2. There exists a bijection

$$\iota\colon \Pi_\phi^0 \xrightarrow{1:1} \widehat{A_\phi^+},\ [\pi^0] \mapsto \iota([\pi^0]).$$

Aim of this section

- Langlands program focuses only on connected reductive groups.
- However, reps. of disconnected groups arise naturally in various context.
- For example, in the theory of theta correspondence, irr. reps. of $\mathrm{Sp}(2m, F)$ and of $\mathrm{O}(2n, F)$ correspond to each other.

Goal of this section

Explain LLC for full (disconnected) orthogonal group $\mathrm{O}(m)$.

Arthur, Mœgline and Heiermann discussed the LLC for $\mathrm{O}(m)$.

Note $\mathrm{O}(2n+1) = \mathrm{SO}(2n+1) \times \{\pm 1\}$. Define

$$\Phi(\mathrm{O}(2n+1)) = \Phi(\mathrm{SO}(2n+1)) \times \{\pm 1\}.$$

LLC for $\mathrm{O}(2n+1)$ is given as follows:

$$\begin{array}{ccc}
\{\pm 1\}^{\wedge} & \xrightarrow{\ \sim\ } & \{\pm 1\} \ni b \\
\uparrow{\scriptstyle \text{central character}} & & \uparrow \\
\Pi_\phi^b \subset \mathrm{Irr}(\mathrm{O}(2n+1, F)) & \dashrightarrow\!\!\!\rightarrow & \Phi(\mathrm{O}(2n+1)) \ni (\phi, b) \\
\downarrow{\scriptstyle \mathrm{Res}} & & \downarrow \\
\Pi_\phi^0 \subset \mathrm{Irr}(\mathrm{SO}(2n+1, F)) & \twoheadrightarrow & \Phi(\mathrm{SO}(2n+1)) \ni \phi,
\end{array}$$

and $\iota \colon \Pi_\phi^b \xrightarrow{1:1} \widehat{A_\phi}$ is defined by $\iota(\sigma) = \iota(\sigma | \mathrm{SO}(2n+1, F))$.

L-parameter for $\mathrm{O}(2n)$

In fact, Arthur gave LLC for $\mathrm{O}(2n)$, and deduced Weak LLC for $\mathrm{SO}(2n)$. Put

$$\begin{aligned}\Phi(\mathrm{O}(2n)) &:= \Phi(\mathrm{SO}(2n))/\sim \\ &= \{\phi\colon WD_F \to \mathrm{O}(2n,\mathbb{C}) \mid \det(\phi) = \omega_{E/F}\}.\end{aligned}$$

Recall

- $A_\phi = \pi_0(\mathrm{Cent}(\mathrm{Im}(\phi), \mathrm{O}(2n,\mathbb{C}))/\{\pm 1\})(\cong (\mathbb{Z}/2\mathbb{Z})^r)$.
- $A_\phi^+ = \pi_0(\mathrm{Cent}(\mathrm{Im}(\phi), \mathrm{SO}(2n,\mathbb{C}))/\{\pm 1\}) = \mathcal{S}_\varphi$.

Then

$$1 \longrightarrow A_\phi^+ \longrightarrow A_\phi \xrightarrow{\det} \mathbb{Z}/2\mathbb{Z} = \{0,1\}.$$

LLC for $\mathrm{O}(2n)$ (Arthur)

1. There exist surjections

$$\begin{array}{ccc} \Pi_\phi \subset \mathrm{Irr}(\mathrm{O}(2n,F)) & \twoheadrightarrow & \Phi(\mathrm{O}(2n)) \ni \phi \\ \downarrow{\scriptstyle \mathrm{Res}} & & \| \\ \Pi_\phi^0 \subset \mathrm{Irr}(\mathrm{SO}(2n,F))/\mathrm{O}(2n,F) & \twoheadrightarrow & \Phi(\mathrm{SO}(2n))/\sim . \end{array}$$

In particular, $\pi \in \Pi_\phi \iff \pi \otimes \det \in \Pi_\phi$.

2. There exist bijections

$$\begin{array}{ccc} \Pi_\phi & \xrightarrow{1:1} & \widehat{A_\phi} \\ \downarrow & & \downarrow \\ \Pi_\phi^0 & \xrightarrow{1:1} & \widehat{A_\phi^+}. \end{array}$$

Property of LLC for $\mathrm{O}(2n)$

There is a special property of LLC for $\mathrm{O}(2n)$.

Proposition

The following are equivalent:

- $\phi \in \Phi(\mathrm{O}(2n))$ contains an irreducible orthogonal odd-dimensional representation of WD_F;
- $[A_\phi : A_\phi^+] = 2$;
- there exists $\pi \in \Pi_\phi$ such that $\pi \otimes \det \not\cong \pi$;
- any $\pi \in \Pi_\phi$ satisfies that $\pi \otimes \det \not\cong \pi$.

Intertwining operator

Let $P = MN \subset \mathrm{O}(2n)$ be a parabolic subgroup with $M \cong \mathrm{GL}_k \times \mathrm{O}(2n_0)$.
If $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{O}(2n_0, F))$ and $\tau \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{GL}_k(F))$,
the induced representation $\mathrm{Ind}_{P(F)}^{\mathrm{O}(2n,F)}(\tau \otimes \pi_0)$ decomposes into
a direct sum of irr. reps. of $\mathrm{O}(2n, F)$:

$$\mathrm{Ind}_{P(F)}^{\mathrm{O}(2n,F)}(\tau \otimes \pi_0) \cong \bigoplus_{\pi} \pi.$$

If τ is self-dual, Arthur defined a normalized self-dual intertwining operator

$$R(w, \tau \otimes \pi_0) \colon \mathrm{Ind}_{P(F)}^{\mathrm{O}(2n,F)}(\tau \otimes \pi_0) \to \mathrm{Ind}_{P(F)}^{\mathrm{O}(2n,F)}(\tau \otimes \pi_0).$$

Intertwining relation

Let

- $\phi_0 \in \Phi(\mathrm{O}(2n_0))$ such that $\pi_0 \in \Pi_{\phi_0}$;
- $\phi_1 \colon WD_F \to \mathrm{GL}_k(\mathbb{C})$ corresponds to τ via LLC for GL_k.

If we put $\phi = \phi_1 \oplus \phi_0 \oplus \phi_1^\vee \in \Phi(\mathrm{O}(2n))$, then $A_{\phi_0} \hookrightarrow A_\phi$.
If ϕ_1 is orthogonal, it define an element $a \in A_\phi$.

Theorem (Arthur)

1. $\Pi_\phi = \{\pi \subset \mathrm{Ind}_{P(F)}^{\mathrm{O}(2n,F)}(\tau \otimes \pi_0) \mid \pi_0 \in \Pi_{\phi_0}\}$;
2. $\iota(\pi)|A_{\phi_0} = \iota(\pi_0)$ if $\pi \subset \mathrm{Ind}_{P(F)}^{\mathrm{O}(2n,F)}(\tau \otimes \pi_0)$;
3. (Intertwining relation)

$$R(w, \tau \otimes \pi_0)|\pi = \iota(\pi)(a) \cdot \mathrm{id}_\pi.$$

How to compute $\iota(\pi)$

Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{O}(2n))$ and $\phi \in \Phi(\mathrm{O}(2n))$ such that $\pi \in \Pi_\phi$.
An element $a \in A_\phi$ defines an orthogonal rep. ϕ_1 contained in ϕ.
Let $\tau \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{GL}_k(F))$ correspond to ϕ_1.
Then by the above theorem, we have:

Propositioin

- $\mathrm{Ind}_{P(F)}^{\mathrm{O}(2n+2k,F)}(\tau \otimes \pi)$ is irreducible;
- $R(w, \tau \otimes \pi) = \iota(\pi)(a) \cdot \mathrm{id}.$

Hence, if one could understand $R(w, \tau \otimes \pi)$ more explicitly,
one could compute $\iota(\pi)(a)$.

1 Local Langlands correspondence (LLC)

2 Local Gross–Prasad conjecture (GP)

3 Arthur's multiplicity formula

Multiplicity one theorem

Now suppose that there exists an inclusion $\mathrm{O}(2n) \hookrightarrow \mathrm{O}(2n+1)$.
Then we have a diagonal map

$$\Delta\colon \mathrm{O}(2n) \to \mathrm{O}(2n) \times \mathrm{O}(2n+1).$$

Multiplicity one theorem

1. (Aizenbud–Gourevitch–Rallis–Schiffmann)
For $\pi \in \mathrm{Irr}(\mathrm{O}(2n,F))$ and $\tau \in \mathrm{Irr}(\mathrm{O}(2n+1,F))$,
$$\dim \mathrm{Hom}_{\Delta \mathrm{O}(2n,F)}(\pi \otimes \tau, \mathbb{C}) \leq 1.$$
2. (Waldspurger) For $\pi^0 \in \mathrm{Irr}(\mathrm{SO}(2n,F))$ and $\tau^0 \in \mathrm{Irr}(\mathrm{SO}(2n+1,F))$,
$$\dim \mathrm{Hom}_{\Delta \mathrm{SO}(2n,F)}(\pi^0 \otimes \tau^0, \mathbb{C}) \leq 1.$$

Local Gross–Prasad conjecture

Local Gross–Prasad conjecture determines precisely when

$$\mathrm{Hom}_{\Delta \mathrm{SO}(2n,F)}(\pi^0 \otimes \tau^0, \mathbb{C}) \neq 0.$$

Let $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}(2n))/\sim$ and $\phi' \in \Phi_{\mathrm{temp}}(\mathrm{SO}(2n+1))$
(i.e., $\phi(W_F)$ and $\phi'(W_F)$ are bounded).
When $\varepsilon(1/2, \phi \otimes \phi', \psi) = 1$, Gross–Prasad defined characters

$$\chi_{\phi'} \colon A_\phi \to \{\pm 1\} \quad \text{and} \quad \chi_\phi \colon A_{\phi'} \to \{\pm 1\}$$

in terms of root numbers.

Gross–Prasad conjecture for $\mathrm{SO}(2n) \times \mathrm{SO}(2n+1)$

Note that $\dim \mathrm{Hom}_{\Delta \mathrm{SO}(2n,F)}(\pi^0 \otimes \tau^0, \mathbb{C})$ depends only on $([\pi^0], \tau^0) \in \mathrm{Irr}(\mathrm{SO}(2n,F))/\mathrm{O}(2n,F) \times \mathrm{Irr}(\mathrm{SO}(2n+1,F))$.

GP for $\mathrm{SO}(2n) \times \mathrm{SO}(2n+1)$ (proved by Waldspurger)

Let $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}(2n))/\sim$ and $\phi' \in \Phi_{\mathrm{temp}}(\mathrm{SO}(2n+1))$
such that $\varepsilon(1/2, \phi \otimes \phi', \psi) = 1$.
Then there exists a unique pair $([\pi^0], \tau^0) \in \Pi^0_\phi \times \Pi^0_{\phi'}$ such that

$$\mathrm{Hom}_{\Delta \mathrm{SO}(2n,F)}(\pi^0 \otimes \tau^0, \mathbb{C}) \neq 0.$$

Moreover,

$$\iota([\pi^0]) = \chi_{\phi'}|A^+_\phi \quad \text{and} \quad \iota(\tau^0) = \chi_\phi.$$

Gross–Prasad conjecture for $\mathrm{O}(2n) \times \mathrm{O}(2n+1)$

We formulated and proved GP for orthogonal groups.

GP for $\mathrm{O}(2n) \times \mathrm{O}(2n+1)$ (formulated and proved by A.-Gan)

Let $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(2n))$ and $(\phi', b) \in \Phi_{\mathrm{temp}}(\mathrm{O}(2n+1))$ such that $\varepsilon(1/2, \phi \otimes \phi', \psi) = 1$.
Then there exists a unique pair $(\pi, \tau) \in \Pi_\phi \times \Pi_{\phi'}^b$ such that

$$\mathrm{Hom}_{\Delta\mathrm{O}(2n,F)}(\pi \otimes \tau, \mathbb{C}) \neq 0.$$

Moreover,

$$\iota(\pi)(a) = \chi_{\phi'}(a) \cdot b^{\det(a)} \text{ for } a \in A_\phi \quad \text{and} \quad \iota(\tau) = \chi_\phi.$$

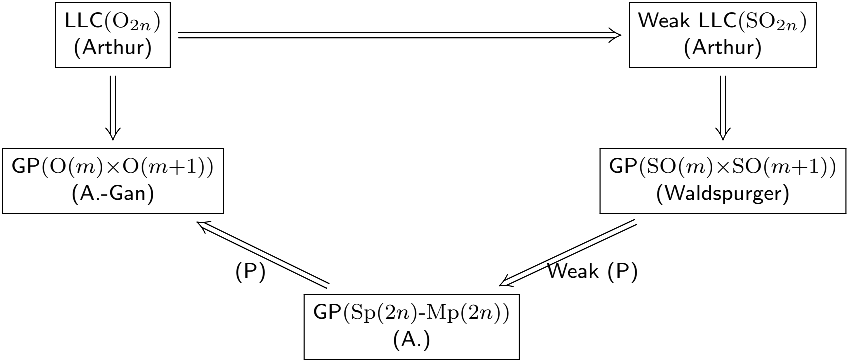
Recall that $1 \longrightarrow A_\phi^+ \longrightarrow A_\phi \xrightarrow{\det} \mathbb{Z}/2\mathbb{Z} = \{0, 1\}.$

Remarks

- In fact, we formulated GP for $\mathrm{O}(m) \times \mathrm{O}(m+1)$ in terms of Vogan L-packets (i.e., for pure inner forms).
  Also, we proved it under assuming *the intertwining relation*.
- When $m = 2$ or 3, there are results of D.Prasad, which are compatible with GP for $\mathrm{O}(m) \times \mathrm{O}(m+1)$.
- A refine version of the global Gross–Prasad conjecture (Ichino–Ikeda conjecture) for orthogonal groups is formulated by H. Xue.

Outline of proof of GP for $\mathrm{O}(m) \times \mathrm{O}(m+1)$

- First, we show D. Prasad's conjecture (P), which describes local theta correspondences in terms of LLC comparing intertwining operators and using the intertwining relation.
- The following is a summary:



A-parameters

From now, let k be a number field, and $\mathbb{A}$ be the adele ring of k. Suppose that $\mathrm{O}(2n)$ is defined and quasi-split over k.

Definition

A global discrete A-parameter for $\mathrm{O}(2n)$ and $\mathrm{SO}(2n)$ is a formal sum

$$\Sigma = \Sigma_1[d_1] \boxplus \cdots \boxplus \Sigma_l[d_l]$$

such that

- Σ_i irr. autom. cusp. unitary rep. of $\mathrm{GL}_{m_i}(\mathbb{A})$;
- $d_i \in \mathbb{Z}_{>0}$ such that $m_1 d_1 + \cdots + m_l d_l = 2n$;
- if $i \neq j$ and $\Sigma_i \cong \Sigma_j$, then $d_i \neq d_j$;
- additional conditions.

A-packets

- We denote the set of global discrete A-parameters by

$$\Psi_2(\mathrm{O}(2n)) = \Psi_2(\mathrm{SO}(2n))/\sim.$$

- Put $\Psi_{2,\mathrm{temp}}(\mathrm{O}(2n)) = \{\boxplus_{i=1}^{l} \Sigma_i[d_i] \mid d_i = 1\}$ to be the set of tempered A-parameters.
- For $\Sigma \in \Psi_2(\mathrm{O}(2n))$, Arthur constructed a global A-packet Π_Σ, which is a multiset of $\mathrm{Irr}(\mathrm{O}(2n, \mathbb{A}))$.
- Let Π_Σ^0 be the image of Π_Σ under $\mathrm{Res}\colon \mathrm{Irr}(\mathrm{O}(2n, \mathbb{A})) \to \mathrm{Irr}(\mathrm{SO}(2n, \mathbb{A}))/\mathrm{O}(2n, \mathbb{A})$.
- If $\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{O}(2n))$, Π_Σ and Π_Σ^0 are multiplicity-free.

Additional notation

Let

- $\mathcal{H}(\mathrm{SO}(2n))$ global Hecke algebra of $\mathrm{SO}(2n, \mathbb{A})$;
- $\mathcal{H}(\mathrm{O}(2n))$ global Hecke algebra of $\mathrm{O}(2n, \mathbb{A})$;
- $\mathcal{A}_2(\mathrm{SO}(2n)) = \{L^2\text{-autom. forms on } \mathrm{SO}(2n, \mathbb{A})\} \curvearrowleft \mathcal{H}(\mathrm{SO}(2n))$.
- $\mathcal{A}_2(\mathrm{O}(2n)) = \{L^2\text{-autom. forms on } \mathrm{O}(2n, \mathbb{A})\} \curvearrowleft \mathcal{H}(\mathrm{O}(2n))$.

Fix $\epsilon = (\epsilon_v)_v \in \mathrm{O}(2n, k) \subset \mathrm{O}(2n, \mathbb{A})$ such that $\det(\epsilon) = -1$, and put $\mathcal{H}^\epsilon(\mathrm{SO}(2n))$ to be the subalgebra of $\mathcal{H}(\mathrm{SO}(2n))$ of functions which are invariant under each ϵ_v.

Arthur's multiplicity formula for $\mathrm{SO}(2n)$

Arthur described $\mathcal{A}_2(\mathrm{SO}(2n))$ in terms of A-packets.

Arthur's multiplicity formula for $\mathrm{SO}(2n)$

For $\Sigma \in \Psi_2(\mathrm{SO}(2n))/\sim$, there is a subset $\Pi_\Sigma^0(\varepsilon_\Sigma^0) \subset \Pi_\Sigma^0$ such that

$$\mathcal{A}_2(\mathrm{SO}(2n)) = \bigoplus_{\Sigma \in \Psi_2(\mathrm{SO}(2n))/\sim} \bigoplus_{[\pi^0] \in \Pi_\Sigma^0(\varepsilon_\Sigma^0)} m_\Sigma[\pi^0]$$

as $\mathcal{H}^\epsilon(\mathrm{SO}(2n))$-modules. Here,

$$m_\Sigma = \begin{cases} 1 & \text{if } \Sigma = \boxplus_{i=1}^{l} \Sigma_i[d_i] \text{ such that } m_i d_i \text{ is odd for some } i, \\ 2 & \text{otherwise.} \end{cases}$$

Arthur's multiplicity formula for $\mathrm{O}(2n)$

We formulated and proved Arthur's multiplicity formula for $\mathrm{O}(2n)$.

Arthur's multiplicity formula for $\mathrm{O}(2n)$ (A.-Gan)

For $\Sigma \in \Psi_2(\mathrm{O}(2n))$, there is a subset $\Pi_\Sigma(\varepsilon_\Sigma) \subset \Pi_\Sigma$ such that

$$\mathcal{A}_2(\mathrm{O}(2n)) = \bigoplus_{\Sigma \in \Psi_2(\mathrm{O}(2n))} \bigoplus_{\pi \in \Pi_\Sigma(\varepsilon_\Sigma)} \pi$$

as $\mathcal{H}(\mathrm{O}(2n))$-modules.

Hence the tempered spectrum

$$\mathcal{A}_{2,\mathrm{temp}}(\mathrm{O}(2n)) = \bigoplus_{\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{O}(2n))} \bigoplus_{\pi \in \Pi_\Sigma(\varepsilon_\Sigma)} \pi$$

is multiplicity-free.

Outline of proof

- Note that $\epsilon \in \mathrm{O}(2n, k)$ acts on $\mathcal{A}_2(\mathrm{SO}(2n))$ by conjugation.
- We regard $\mathcal{A}_2(\mathrm{SO}(2n))$ as an $(\mathcal{H}(\mathrm{SO}(2n)), \epsilon)$-module.
- Arthur also described $\mathcal{A}_2(\mathrm{SO}(2n))$ as $(\mathcal{H}(\mathrm{SO}(2n)), \epsilon)$-modules in terms of A-packets.
- Note that if π^0 is automorphic (i.e., occurs in $\mathcal{A}_2(\mathrm{SO}(2n))$), then its conjugate $(\pi^0)^\epsilon$ by ϵ is also automorphic.
- Translate Arthur's results via the restriction map

$$\mathrm{Res}\colon \mathcal{A}_2(\mathrm{O}(2n)) \to \mathcal{A}_2(\mathrm{SO}(2n)),$$

which is a *surjective* $(\mathcal{H}(\mathrm{SO}(2n)), \epsilon)$-homomorphism.

What is m_Σ ?

Let $\Sigma = \boxplus_{i=1}^{l} \Sigma_i[d_i] \in \Psi_2(\mathrm{SO}(2n))/\sim = \Psi_2(\mathrm{O}(2n))$.
Suppose $[\pi^0] \in \Pi_\Sigma^0$ and π^0 is automorphic (i.e., occurs in $\mathcal{A}_2(\mathrm{SO}(2n))$).
There are three cases as follows:

(A) If $m_i d_i$ is odd for some i, then $m_\Sigma = 1$ and $\pi^0 \cong (\pi^0)^\epsilon$.
In this case, there are many extensions of π^0 to $\mathrm{O}(2n, \mathbb{A})$, and exactly half of them are automorphic;

(B) If $m_i d_i$ is even for any i and $\pi^0 \not\cong (\pi^0)^\epsilon$, then $m_\Sigma = 2$.
In this case, there is $\pi \subset \mathcal{A}_2(\mathrm{O}(2n))$ such that $\mathrm{Res}(\pi) = \pi^0 \oplus (\pi^0)^\epsilon$;

(C) If $m_i d_i$ is even for any i and $\pi^0 \cong (\pi^0)^\epsilon$, then $m_\Sigma = 2$.
In this case, there are $\pi_1, \pi_2 \subset \mathcal{A}_2(\mathrm{O}(2n, \mathbb{A}))$ such that $\pi_1|\mathrm{SO}(2n, \mathbb{A}) \cong \pi^0 \cong \pi_2|\mathrm{SO}(2n, \mathbb{A})$ but $\mathrm{Res}(\pi_1) \neq \mathrm{Res}(\pi_2)$ as subspaces of $\mathcal{A}_2(\mathrm{SO}(2n, \mathbb{A}))$.

Thank you

Thank you so much for your kind attention.