Alpha-determinants and the Alon-Tarsi conjecture

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New Developments in Representation Theory

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内容

- Introduction to Alpha-determinants
- Cyclic $U(\mathfrak{gl}_n)$-modules generated by alpha-determinant
- Wreath determinants for rectangular matrices of particular types
- The Alon-Tarsi conjecture on Latin squares

Alpha-determinant

Definition (Alpha-determinant)

Let α be a parameter. For $A = (a_{ij}) \in M_N$, define

$$\det_\alpha A := \sum_{\sigma \in \mathfrak{S}_N} \alpha^{\nu(\sigma)} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(N)N},$$

where $\nu(\sigma) = N - (\text{number of disjoint cycles in } \sigma)$.

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where $\nu(\sigma) = N -$ (number of disjoint cycles in σ).

Example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 1 & 3 & 5 & 2 \end{pmatrix} = (1\ 4\ 3)(2\ 6)(5) \in \mathfrak{S}_6$$

$\implies \nu(\sigma) = 6 - 3 = 3$

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where $\nu(\sigma) = N -$ (number of disjoint cycles in σ).

Example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 1 & 3 & 5 & 2 & 7 \end{pmatrix} = (1\ 4\ 3)(2\ 6)(5)(7) \in \mathfrak{S}_7$$

$$\Longrightarrow \nu(\sigma) = (6+1) - (3+1) = 3$$

Example

$$\begin{aligned}
\sigma = e = (1)(2)(3) &\implies \nu(\sigma) = 3 - 3 = 0 \\
\sigma = (1\,2\,3), (1\,3\,2) &\implies \nu(\sigma) = 3 - 1 = 2 \\
\sigma = (2\,3)(1), (1\,2)(3), (1\,3)(2) &\implies \nu(\sigma) = 3 - 2 = 1
\end{aligned}$$

$$\begin{aligned}
\det_\alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}a_{22}a_{33} + \alpha^2 a_{12}a_{23}a_{31} + \alpha^2 a_{13}a_{21}a_{32} \\
&\quad + \alpha a_{11}a_{23}a_{32} + \alpha a_{12}a_{21}a_{33} + \alpha a_{13}a_{22}a_{31}
\end{aligned}$$

Example

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\sigma = e = (1)(2)(3) \quad &\Longrightarrow \quad \nu(\sigma) = 3 - 3 = 0 \\
\sigma = (1\,2\,3), (1\,3\,2) \quad &\Longrightarrow \quad \nu(\sigma) = 3 - 1 = 2 \\
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If $\alpha = -1$, then $\det_\alpha = \det$

$$\det_\alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Example

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\end{aligned}$$

If $\alpha = 1$, then $\det_\alpha = \mathrm{per}$

$$\det_\alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$+ a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}$$

Basic properties

- Multilinear with respect to rows and columns
- Invariant under transposition $A \mapsto {}^tA$
- Not multiplicative in general (multiplicative $\iff \alpha = -1$)
- $\det_\alpha(AP(\sigma)) = \det_\alpha(P(\sigma)A)$ for $A \in M_n$ and $\sigma \in \mathfrak{S}_n$, where $P(\sigma) = (\delta_{i\sigma(j)})$ is the permutation matrix of σ
- Laplace expansion

Laplace expansion

$$\det_\alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + \alpha^2 a_{12}a_{23}a_{31} + \alpha^2 a_{13}a_{21}a_{32}$$

$$+ \alpha a_{11}a_{23}a_{32} + \alpha a_{12}a_{21}a_{33} + \alpha a_{13}a_{22}a_{31}$$

$$= a_{11} \det_\alpha \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} + \alpha a_{12} \det_\alpha \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

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Laplace expansion

Example

For the all-one matrix $\mathbf{1}_n$ of size n, we have

$$\begin{aligned}\det_\alpha \mathbf{1}_n &= \det_\alpha \mathbf{1}_{n-1} + \overbrace{\alpha \det_\alpha \mathbf{1}_{n-1} + \cdots + \alpha \det_\alpha \mathbf{1}_{n-1}}^{n-1} \\ &= (1 + (n-1)\alpha) \det_\alpha \mathbf{1}_{n-1}\end{aligned}$$

Laplace expansion

Example

For the all-one matrix $\mathbf{1}_n$ of size n, we have

$$\begin{aligned}\det_\alpha \mathbf{1}_n &= \det_\alpha \mathbf{1}_{n-1} + \overbrace{\alpha \det_\alpha \mathbf{1}_{n-1} + \cdots + \alpha \det_\alpha \mathbf{1}_{n-1}}^{n-1} \\ &= (1+(n-1)\alpha) \det_\alpha \mathbf{1}_{n-1} \\ &= (1+\alpha)(1+2\alpha)\cdots(1+(n-1)\alpha)\end{aligned}$$

History

- Vere-Jones (1988) introduce the alpha-determinant (but he called it "α-permanent) to treat the probability density functions of multivariate binomial and negative binomial distributions uniformly.
- Shirai and Takahashi (2003) utilized the alpha-determinant to introduce a parametric family of point processes which generalize the fermion and boson point processes.
- Matsumoto and Wakayama (2006) studied the $U(\mathfrak{gl}_n)$-cyclic module generated by $\det_\alpha X$ $(X = (x_{ij}))$.
- Generalization and quantum group analog of the study by Matsumoto and Wakayama.

Cyclic $U(\mathfrak{gl}_n)$-modules generated by $\det_\alpha X$

The algebra $\mathcal{P}(M_n)$ of polynomial functions on M_n becomes a $U(\mathfrak{gl}_n)$-module by

$$E_{ij} \cdot f(X) = \sum_{s=1}^{n} x_{is} \frac{\partial f}{\partial x_{js}}(X).$$

Consider the cyclic module $V_n(\alpha) := U(\mathfrak{gl}_n) \cdot \det_\alpha(X)$ $(X = (x_{ij}))$. Notice that

$$V_n(-1) = \mathbb{C} \cdot \det X \cong \wedge^n(\mathbb{C}^n), \qquad V_n(1) \cong S^n(\mathbb{C}^n)$$

are both **irreducible**.

Irreducible decomposition of $V_n(\alpha)$

S. Matsumoto and M. Wakayama (2006) proved that

$$V_n(\alpha) \cong \bigoplus_{\substack{\lambda \vdash n \\ f_\lambda(\alpha) \neq 0}} \left(\boldsymbol{E}_n^\lambda\right)^{\oplus f^\lambda}$$

where

- we identify highest weights and partitions,
- $\boldsymbol{E}_n^\lambda$: irreducible $U(\mathfrak{gl}_n)$-module with highest weight λ,
- f^λ: number of standard tableaux of λ,
- $f_\lambda(x) = \prod_{(i,j)\in\lambda}\bigl(1-(j-i)x\bigr)$

Example: $n = 3$

$$
\begin{aligned}
f_{(3)}(x) &= (1+x)(1+2x), \\
f_{(21)}(x) &= (1+x)(1-x), \\
f_{(111)}(x) &= (1-x)(1-2x)
\end{aligned}
$$

$$
\Longrightarrow \quad V_3(\alpha) \cong \begin{cases}
\boldsymbol{E}_3^{(3)} & \alpha = 1 \\
\boldsymbol{E}_3^{(3)} \oplus \left(\boldsymbol{E}_3^{(21)}\right)^{\oplus 2} & \alpha = \frac{1}{2} \\
\left(\boldsymbol{E}_3^{(21)}\right)^{\oplus 2} \oplus \boldsymbol{E}_3^{(111)} & \alpha = -\frac{1}{2} \\
\boldsymbol{E}_3^{(111)} & \alpha = -1 \\
\boldsymbol{E}_3^{(3)} \oplus \left(\boldsymbol{E}_3^{(21)}\right)^{\oplus 2} \oplus \boldsymbol{E}_3^{(111)} & \text{otherwise}
\end{cases}
$$

Sketch of proof

We have

$$\alpha^{\nu(\sigma)} = \frac{1}{n!} \sum_{\lambda \vdash n} f^{\lambda} f_{\lambda}(\alpha) \chi^{\lambda}(\sigma) \qquad (\sigma \in \mathfrak{S}_n),$$

where χ^{λ} is the irreducible character of $\mathfrak{S}_n$ corresponding to λ.

Sketch of proof

$$\det_\alpha X = \frac{1}{n!} \sum_{\lambda \vdash n} f^\lambda f_\lambda(\alpha) \operatorname{Imm}^\lambda(X),$$

where

$$\operatorname{Imm}^\lambda(X) = \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma)\, x_{1\sigma(1)} \cdots x_{n\sigma(n)}$$

is the **immanant** of X associated to λ. Use

$$U(\mathfrak{gl}_n) \cdot \operatorname{Imm}^\lambda(X) \cong (\boldsymbol{E}_n^\lambda)^{\oplus f^\lambda}.$$

Generalization

$$V_{n,l}(\alpha) := U(\mathfrak{gl}_n) \cdot (\det_\alpha X)^l$$

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$$\Longrightarrow \; V_{n,l}(\alpha) = \bigoplus_{\substack{\lambda \vdash nl \\ l(\lambda) \le n}} (\boldsymbol{E}_n^\lambda)^{\oplus \operatorname{rk} F_{n,l}^\lambda(\alpha)},$$

where $F_{n,l}^\lambda(\alpha)$ is a certain square matrix depending on α, and rk is the matrix rank.

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where $F_{n,l}^\lambda(\alpha)$ is a certain square matrix depending on α, and rk is the matrix rank. When $l = 1$, we have

$$F_{n,1}^\lambda(\alpha) = f_\lambda(\alpha) I_{f^\lambda} \; \Longrightarrow \; \operatorname{rk} F_{n,1}^\lambda(\alpha) = \begin{cases} f^\lambda & f_\lambda(\alpha) \neq 0 \\ 0 & f_\lambda(\alpha) = 0 \end{cases}$$

- If λ is of **hook-type**, then $F^{\lambda}_{n,l}(\alpha) = f^{\lambda}_{n,l}(\alpha)I$ for some polynomial $f^{\lambda}_{n,l}(x)$.
- If $n = 2$, then $F^{\lambda}_{2,l}(\alpha)$ is 1×1 (i.e. a polynomial) and is explicitly given by

$$F^{(2l-s,s)}_{2,l}(\alpha) = (1+\alpha)^{l-s} {}_2F_1(-s, l-s+1; -l; -\alpha)$$

for $s = 0, 1, \ldots, l$, where ${}_2F_1(a, b; c; x)$ is the Gaussian hypergeometric series.

Permanent cyclic modules

We conjectured that

$$V_{n,l}(1) = U(\mathfrak{gl}_n) \cdot (\operatorname{per} X)^l \cong S^l(S^n(\mathbb{C}^n)).$$

When $n = 2$, we can show that

$$F_{2,l}^{(2l-s,s)}(1) \neq 0 \iff s \text{ is even},$$

which implies

$$V_{2,l}(1) \cong \bigoplus_{\substack{0 \le s \le l \\ 2 \mid s}} \boldsymbol{E}_2^{(2l-s,s)} \cong S^l(S^2(\mathbb{C}^2)).$$

Weak alternating property

The ordinary determinant is alternating in rows and columns:
If distinct two rows or columns in A are identical, then $\det A = 0$.

When $\alpha = -1/k$ for a positive integer k, $\det_\alpha$ possesses a similar and weaker property to this.

Weak alternating property

Let $k < n$. For $A \in M_n$,

$$\sum_{\sigma \in \mathfrak{S}_{k+1}} \det_\alpha(AP(\sigma)) = \sum_{\sigma \in \mathfrak{S}_{k+1}} \det_\alpha(P(\sigma)A)$$

is divisible by $(1+\alpha)(1+2\alpha)\cdots(1+k\alpha)$.

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If first $k+1$ columns (or rows) in A are identical, then

$$\det_\alpha A = \frac{1}{(k+1)!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \det_\alpha(AP(\sigma))$$

Weak alternating property

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is divisible by $(1+\alpha)(1+2\alpha)\cdots(1+k\alpha)$.

If first $k+1$ columns (or rows) in A are identical, then

$$\det_{\boldsymbol{-1/k}} A = \frac{1}{(k+1)!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \det_{\boldsymbol{-1/k}}(AP(\sigma)) = \mathbf{0}$$

Wreath determinant

Definition (k-wreath determinant)

For $A \in M_{n,kn}$, define

$$\mathrm{wrdet}_k A := \det_{-1/k}(A \otimes \mathbf{1}_{k,1}),$$

where $\mathbf{1}_{k,1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is the $k \times 1$ all-one matrix.

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \otimes \mathbf{1}_{2,1} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

Example

$$\operatorname{wrdet}_2 \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} = \det_{-1/2} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

$$= \frac{a_1 a_2 b_3 b_4 + b_1 b_2 a_3 a_4}{4} - \frac{(a_1 b_2 + b_1 a_2)(a_3 b_4 + b_3 a_4)}{8}$$

Basic properties of the wreath determinant

Proposition

Let $f\colon M_{n,kn} \ni A \mapsto \mathrm{wrdet}_k A \in \mathbb{C}$. Then

1. f is multilinear in column vectors,
2. $f(QA) = (\det Q)^k f(A) \quad (Q \in GL_n)$,
3. $f(AP(\sigma)) = f(A) \quad (\sigma \in \mathcal{K}_{n,k})$.

Here

$$\mathcal{K}_{n,k} := \{\sigma \in \mathfrak{S}_{kn} \mid \sigma\Omega_i = \Omega_i \ (1 \le i \le n)\},$$

$$\Omega_i = \{(i-1)k+1, (i-1)k+2, \dots, ik\} \quad (1 \le i \le n)$$

When $k = n = 2$,

$$\Omega_1 = \{1,2\}, \quad \Omega_2 = \{3,4\}, \quad \mathcal{K}_{2,2} = \{e, (1\,2), (3\,4), (1\,2)(3\,4)\}$$

Proof.

1. Inherit from $\det_{-1/k}$.
2. Enough to prove when Q is a fundamental matrix. It is clear when Q is diagonal. When $Q = I_n + aE_{ij}$ $(i \neq j)$, use multilinearity and "weak alternating property" of $\det_{-1/k}$ to prove $f(QA) = f(A)$.
3. By definition, we have

$$\begin{aligned} \operatorname{wrdet}_k AP(\sigma) &= \det_{-1/k}((AP(\sigma) \otimes \mathbf{1}_{k,1})) \\ &= \det_{-1/k}((A \otimes \mathbf{1}_{k,1})P(\sigma)) \\ &= \det_{-1/k}(P(\sigma)(A \otimes \mathbf{1}_{k,1})) \\ &= \det_{-1/k}(A \otimes \mathbf{1}_{k,1}) = \operatorname{wrdet}_k A \end{aligned}$$

for $\sigma \in \mathcal{K}_{n,k}$.

□

Characterization of the wreath determinant

Proposition

If $f\colon M_{n,kn} \to \mathbb{C}$ satisfies the conditions

1. f is multilinear in column vectors,
2. $f(QA) = (\det Q)^k f(A) \quad (Q \in GL_n)$,
3. $f(AP(\sigma)) = f(A) \quad (\sigma \in \mathcal{K}_{n,k})$,

then f equals wrdet_k up to a constant multiple.

Proof.

By a combinatorial discussion, we can prove that the coefficient of each monomials in $f(X)$ $(X = (x_{ij}))$ are uniquely determined (up to constant) by the given three conditions. □

Another proof.

We regard $f(X)$ ($X = (x_{ij})_{1\le i\le n, 1\le j\le kn}$) as an element in $\mathcal{P}(M_{n,kn})$. By the Howe duality, we have

$$\mathcal{P}(M_{n,kn}) = \sum_{\lambda} \mathcal{P}^{\lambda}(M_{n,kn}), \quad \mathcal{P}^{\lambda}(M_{n,kn}) \cong \boldsymbol{E}_n^{\lambda} \boxtimes \boldsymbol{E}_{kn}^{\lambda}$$

as a $GL_n \times GL_{kn}$-module. $f(X)$ lives in

$$\mathcal{P}^{(k^n)}(M_{n,kn})^{\mathcal{K}_{n,k}}_{(1,\dots,1)} = \left\{ f(X) \in \mathcal{P}^{(k^n)}(M_{n,kn}) \,\middle|\, \begin{array}{l} f(XT) = \det T\, f(X)\ (T \in \mathbb{T}_{kn}) \\ f(XP(\sigma)) = f(X)\ (\sigma \in \mathcal{K}_{n,k}) \end{array} \right\},$$

whose dimension is $K_{(k^n)(k^n)} = 1$ (Kostka number). □

Zonal spherical function for a rectangular diagram

For $\lambda \vdash kn$, define

$$\omega^{\lambda}(g) := \frac{1}{|\mathcal{K}_{n,k}|} \sum_{h \in \mathcal{K}_{n,k}} \chi^{\lambda}(gh),$$

where χ^{λ} is the irreducible character of $\mathfrak{S}_{kn}$ corresponding to λ.
Especially, we put

$$\omega_{n,k}(g) := \omega^{(k^n)}(g),$$

where $(k^n) = (k, k, \ldots, k) \vdash kn$.

Zonal spherical function for a rectangular diagram

ω^λ is a $\mathcal{K}_{n,k}$-biinvariant function on $\mathfrak{S}_{kn}$:

$$\omega^\lambda(h_1 g h_2) = \omega^\lambda(g) \quad (g \in \mathfrak{S}_{kn}, h_1, h_2 \in \mathcal{K}_{n,k})$$

The function ω^λ is identically zero unless $\lambda \geq (k^n)$, where $\geq$ is the **dominance ordering** among partitions:

$$\lambda \geq \mu \iff \sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i \quad (\forall r \geq 1).$$

Zonal spherical function for a rectangular diagram

For $g \in \mathfrak{S}_{kn}$,

$$m_{ij}(g) := |g\Omega_i \cap \Omega_j| \quad (1 \le i, j \le n)$$

$$\Big(\Omega_i = \{(i-1)k+1, (i-1)k+2, \dots, ik\} \quad (1 \le i \le n)\Big)$$

$$M(g) := \Big(m_{ij}(g)\Big)_{1 \le i,j \le n}$$

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$$M(g) := \Big(m_{ij}(g)\Big)_{1 \le i,j \le n}$$

$$\mathcal{K}_{n,k} g \mathcal{K}_{n,k} = \mathcal{K}_{n,k} g' \mathcal{K}_{n,k} \iff M(g) = M(g')$$

Zonal spherical function for a rectangular diagram

Theorem

$$\omega_{n,k}(g) = \frac{M(g)!}{|\mathcal{K}_{n,k}|} \times \text{the coefficient of } x^{M(g)} \text{ in } (\det X)^k$$
$$= \frac{\mathrm{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(g))}{\mathrm{wrdet}_k(I_n \otimes \mathbf{1}_{1,k})}$$

for $g \in \mathfrak{S}_{kn}$, where

$$X = (x_{ij})_{1 \le i,j \le n},$$

$$x^M = \prod_{i,j=1}^{n} x_{ij}^{m_{ij}}, \qquad M! = \prod_{i,j=1}^{n} m_{ij}! \qquad (M = (m_{ij}))$$

Proof.

$$\omega^{\lambda}(g) = \frac{1}{|\mathcal{K}_{n,k}|} \operatorname{Imm}^{\lambda}((I_n \otimes \mathbf{1}_k)P(g))$$

$$\begin{aligned} \operatorname{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(g)) &= \det_{-1/k}((I_n \otimes \mathbf{1}_k)P(g)) \\ &= \frac{|\mathcal{K}_{n,k}|}{(kn)!} \sum_{\lambda \vdash kn} f^{\lambda} f_{\lambda}(-1/k) \omega^{\lambda}(g). \end{aligned}$$

Proof.

$$\omega^\lambda(g) = \frac{1}{|\mathcal{K}_{n,k}|} \operatorname{Imm}^\lambda((I_n \otimes \mathbf{1}_k)P(g))$$

$$\begin{aligned} \operatorname{wrdet}_k((I_n \otimes \mathbf{1}_{1,k})P(g)) &= \det_{-1/k}((I_n \otimes \mathbf{1}_k)P(g)) \\ &= \frac{|\mathcal{K}_{n,k}|}{(kn)!} \sum_{\lambda \vdash kn} f^\lambda f_\lambda(-1/k)\omega^\lambda(g). \end{aligned}$$

Since $f_\lambda(-1/k) = 0$ if $\lambda_1 > k$, and $\omega^\lambda = 0$ unless $\lambda \geq (k^n)$,

$$\frac{|\mathcal{K}_{n,k}|}{(kn)!} \sum_{\lambda \vdash kn} f^\lambda f_\lambda(-1/k)\omega^\lambda(g) = \frac{|\mathcal{K}_{n,k}|}{(kn)!} f^{(k^n)} f_{(k^n)}(-1/k)\omega_{n,k}(g).$$

We obtain the theorem by taking the ratio. □

Example $(k = n = 2)$

For $g_1 = e, g_2 = (2\,3), g_3 = (1\,4)(2\,3) \in \mathfrak{S}_4$,

$$M(g_1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad M(g_2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M(g_3) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$x^{M(g_1)} = x_{11}^2 x_{22}^2, \quad x^{M(g_2)} = x_{11}x_{12}x_{21}x_{22}, \quad x^{M(g_3)} = x_{12}^2 x_{21}^2,$$

Example ($k = n = 2$)

For $g_1 = e, g_2 = (2\,3), g_3 = (1\,4)(2\,3) \in \mathfrak{S}_4$,

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$$x^{M(g_1)} = x_{11}^2 x_{22}^2, \quad x^{M(g_2)} = x_{11}x_{12}x_{21}x_{22}, \quad x^{M(g_3)} = x_{12}^2 x_{21}^2,$$

$$(\det X)^2 = x_{11}^2 x_{22}^2 - 2x_{11}x_{12}x_{21}x_{22} + x_{12}^2 x_{21}^2$$

$$\begin{aligned} \omega_{2,2}(g_1) &= \frac{2!0!0!2!}{4} \times 1 = 1, \\ \omega_{2,2}(g_2) &= \frac{1!1!1!1!}{4} \times (-2) = -\frac{1}{2}, \\ \omega_{2,2}(g_3) &= \frac{0!2!2!0!}{4} \times 1 = 1 \end{aligned}$$

Example $(k = n = 2)$

$$(I_2 \otimes \mathbf{1}_{1,2})P(g_1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} (= I_2 \otimes \mathbf{1}_{1,2}),$$
$$(I_2 \otimes \mathbf{1}_{1,2})P(g_2) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$
$$(I_2 \otimes \mathbf{1}_{1,2})P(g_3) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Example ($k = n = 2$)

$$(I_2 \otimes \mathbf{1}_{1,2})P(g_1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} (= I_2 \otimes \mathbf{1}_{1,2}),$$

$$(I_2 \otimes \mathbf{1}_{1,2})P(g_2) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$(I_2 \otimes \mathbf{1}_{1,2})P(g_3) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\omega_{2,2}(g_1) = \frac{1/4}{1/4} = 1,$$

$$\omega_{2,2}(g_2) = \frac{-1/8}{1/4} = -\frac{1}{2},$$

$$\omega_{2,2}(g_3) = \frac{1/4}{1/4} = 1$$

Latin squares

A **Latin square** of degree n is an $n \times n$ matrix L whose rows and columns are permutations of $1, 2, \ldots, n$.

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Example

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

1	2	3
3	1	2
2	3	1

1	3	2
2	1	3
3	2	1

2	3	1
3	1	2
1	2	3

3	2	1
2	1	3
1	3	2

2	3	1
1	2	3
3	1	2

2	1	3
3	2	1
1	3	2

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1	3	2
3	2	1
2	1	3

3	1	2
2	3	1
1	2	3

3	2	1
1	3	2
2	1	3

1	2	3
2	3	1
3	1	2

2	1	3
1	3	2
3	2	1

The numeric values of the number $L(n)$ of Latin squares of degree n is known for $n \le 11$ (OEIS `A002860`):

$$
\begin{aligned}
L(1) &= 1, \\
L(2) &= 2, \\
L(3) &= 12, \\
L(4) &= 576, \\
L(5) &= 161280, \\
L(6) &= 812851200, \\
L(7) &= 61479419904000, \\
L(8) &= 108776032459082956800, \\
L(9) &= 5524751496156892842531225600, \\
L(10) &= 9982437658213039871725064756920320000, \\
L(11) &= 776966836171770144107444346734230682311065600000.
\end{aligned}
$$

Parity of Latin squares

A Latin square L of degree n can be written in the form

$$L = 1P(\sigma_1) + 2P(\sigma_2) + \cdots + nP(\sigma_n)$$

$$(\text{with } P(\sigma_1) + P(\sigma_2) + \cdots + P(\sigma_n) = \mathbf{1}_n)$$

for some $\sigma_1, \sigma_2, \ldots, \sigma_n \in \mathfrak{S}_n$. Then we define the **signature** of L by

$$\operatorname{sgn} L := (\operatorname{sgn} \sigma_1)(\operatorname{sgn} \sigma_2) \cdots (\operatorname{sgn} \sigma_n) = \operatorname{sgn}(\sigma_1 \sigma_2 \cdots \sigma_n).$$

We call L **even** when $\operatorname{sgn} L = 1$, and **odd** when $\operatorname{sgn} L = -1$.

Example

$$
L = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} & 2 & & \\ & & & 2 \\ 2 & & & \\ & & 2 & \end{pmatrix}
$$

$$
+ \begin{pmatrix} & & 3 & \\ & 3 & & \\ & & & 3 \\ 3 & & & \end{pmatrix} + \begin{pmatrix} & & & 4 \\ 4 & & & \\ & & 4 & \\ & 4 & & \end{pmatrix}
$$

$$
= 1P((2\,3)) + 2P((1\,3\,4\,2)) + 3P((1\,4\,3)) + 4P((1\,2\,4))
$$

$$
\implies \operatorname{sgn} L = \operatorname{sgn}(2\,3)(1\,3\,4\,2)(1\,4\,3)(1\,2\,4) = +1
$$

Hence L is even.

Example

$$
\mathbf{1}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} & 1 & & \\ & & & 1 \\ 1 & & & \\ & & 1 & \end{pmatrix}
$$

$$
+ \begin{pmatrix} & & 1 & \\ & 1 & & \\ & & & 1 \\ 1 & & & \end{pmatrix} + \begin{pmatrix} & & & 1 \\ 1 & & & \\ & & 1 & \\ & 1 & & \end{pmatrix}
$$

$$
= P((2\,3)) + P((1\,3\,4\,2)) + P((1\,4\,3)) + P((1\,2\,4))
$$

$$
\Longrightarrow \operatorname{sgn} L = \operatorname{sgn}(2\,3)(1\,3\,4\,2)(1\,4\,3)(1\,2\,4) = +1
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1	2	3
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3	2	1
1	3	2
2	1	3

even

2	3	1
3	1	2
1	2	3

3	2	1
2	1	3
1	3	2

3	1	2
1	2	3
2	3	1

1	3	2
3	2	1
2	1	3

1	2	3
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odd

The Alon-Tarsi conjecture

If n is **odd**, then the numbers of even and odd Latin squares of degree n are **equal**:

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$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} \\ 3 & 1 & 2 \end{pmatrix} \longmapsto \begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 3 & 1 & 2 \end{pmatrix}$$

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The Alon-Tarsi conjecture (1992)

If n is **even**, then the numbers of even and odd Latin squares of degree n are **not equal**.

(Origin: the Alon-Tarsi conjecture $\implies$ the line graph of $K_{n,n}$ is n-choosable)

Known results

The Alon-Tarsi conjecture (1992)

If n is **even**, then the numbers of even and odd Latin squares of degree n are **not equal**.

- Drisko (1997) proved the case where $n = p + 1$ (p: odd prime)
- Glynn (2010) proved the case where $n = p - 1$ (p: odd prime)

Wreath determinants and the Alon-Tarsi conjecture

Notice that

$$\det X = \sum_{\sigma} \operatorname{sgn}(\sigma)\, x^{P(\sigma)}.$$

Thus it follows that

$$(\det X)^n = \sum_{\sigma_1,\dots,\sigma_n \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1\sigma_2\cdots\sigma_n) x^{P(\sigma_1)+P(\sigma_2)+\cdots+P(\sigma_n)}$$

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$\therefore$ the coefficient of $x^{\mathbf{1}_n}$ in $(\det X)^n$

$$= \sum_{\substack{\sigma_1,\dots,\sigma_n \in \mathfrak{S}_n \\ P(\sigma_1)+\cdots+P(\sigma_n)=\mathbf{1}_n}} \operatorname{sgn}(\sigma_1\sigma_2\cdots\sigma_n)$$

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Wreath determinants and the Alon-Tarsi conjecture

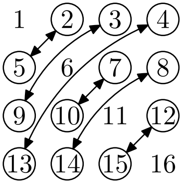
Theorem

The Alon-Tarsi conjecture on Latin squares of degree n is equivalent to each of the followings:

1. the coefficient of $x^{\mathbf{1}_n}$ in $(\det X)^n \neq 0$
2. $\omega_{n,n}(g_n) \neq 0$
3. $\mathrm{wrdet}_n((I_n \otimes \mathbf{1}_{1,n})P(g_n)) \neq 0$

Here $g_n \in \mathfrak{S}_{n^2}$ is defined by $g_n((i-1)n+j) = (j-1)n+i$ $(1 \leq i, j \leq n)$ (which satisfies $M(g_n) = \mathbf{1}_n$).

$$g_4 = (2\ 5)(3\ 9)(4\ 13)(7\ 10)(8\ 14)(12\ 15)$$



Alternative proof of Glynn's result

If $n = p - 1$ for a prime p, then

$$-\frac{1}{n} \equiv 1 \pmod{p}, \qquad \left|\mathfrak{S}_{(n^n)}\right| = (p-1)!^{p-1} \equiv 1 \pmod{p},$$

so it follows that

$$\mathrm{wrdet}_{-1/n}((I_n \otimes \mathbf{1}_{1,n})P(g_n)) = \sum_{\sigma \in \mathfrak{S}_{(n^n)}} \left(-\frac{1}{n}\right)^{\nu(g_n \sigma)}$$

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$$\therefore \ \operatorname{wrdet}_{-1/n}(\mathbb{I}_{n,n} P(g_n)) \neq 0$$

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