Conormal variety on exotic Grassmannian

joint work in progress with Lucas Fresse

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Plan of talk

1. Conormal variety
   Review the properties of conormal varieties
2. Double flag variety for symmetric pair
   - Introduce double flag variety
   - Moment maps, nilpotent varieties, Steinberg theory
3. Exotic Grassmannian and nilpotent variety
   - K-orbits on Exotic Grassmannian
   - Exotic nilpotent variety
4. Some combinatorics
   - Marked Young tableaux & Striped signed Young diagram
   - Correspondence via Young diagrams
5. Orbit correspondence
   - Bijecton of orbits: Hermitian symmetric case
   - General correspondence

Conormal variety: a review

G : algebraic group / $\mathbb{C}$ $\qquad \mathfrak{g} = \mathrm{Lie}(G)$
X : smooth variety $\curvearrowleft G$

T^*X : cotangent bdle (symplectic) $\curvearrowleft G$ by Hamiltonian action

$$\Longrightarrow \exists \text{ moment map } \mu : \begin{array}{ccc} T^*X & \longrightarrow & \mathfrak{g}^* \\ \Cup & & \Cup \\ (x, \xi) & \longmapsto & (z \mapsto \xi(z_x)) \end{array}$$

z_x : vector field at $x \in X$ generated by $z \in \mathfrak{g}$

Definition 2.1

$\mathcal{Y}_X := \mu^{-1}(0) \subset T^*X$: conormal variety

$X/G \ni \mathbb{O}$: G-orbit $\rightsquigarrow T^*_{\mathbb{O}}X = \bigcup_{x \in \mathbb{O}} (T_x\mathbb{O})^{\perp}$: conormal bdle

Lagrangian subvar of dim $= \dim X$

$X/G \ni \mathbb{O}$: G-orbit $\rightsquigarrow$ $T^*_{\mathbb{O}}X$: conormal bdle

Lemma 2.2

$\mathcal{Y}_X = \bigsqcup_{\mathbb{O} \in X/G} T^*_{\mathbb{O}}X$ *(hence the name of conormal variety)*

Proof.

$$(x,\xi) \in \mu^{-1}(0) \iff \mu(x,\xi)(z) = \xi(z_x) = 0 \quad (\forall z \in \mathfrak{g})$$
$$\iff \xi \in (T_x\mathbb{O})^{\perp} \quad (\mathbb{O} := G \cdot x)$$

□

Corollary 2.3

Assume $\#X/G < \infty$.

1. $\mathcal{Y}_X$ *is equi-dimensional of* $\dim X$ *and*
2. $\mathcal{Y}_X = \bigcup_{\mathbb{O} \in X/G} \overline{T^*_{\mathbb{O}}X}$ *gives irred decomposition as an alg variety*

$(\because \quad T^*_{\mathbb{O}}X$: irreducible and $\dim T^*_{\mathbb{O}}X = \dim X)$

Double flag variety — Definition (N-Ochiai [NO11])

G : reductive alg grp $/\mathbb{C}$ $\theta \in \mathrm{Aut}\, G$: involution
$\rightsquigarrow$ $K = G^\theta$: symmetric subgrp ($\fallingdotseq$ $\mathbb{C}$-fication of max cpt subgrp)

Example 3.1

1. type A : $(G, K) = (\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$, $(\mathrm{GL}_n, \mathrm{GL}_p \times \mathrm{GL}_q)$ $(n = p + q)$
2. type C : $(G, K) = (\mathrm{Sp}_{2n}, \mathrm{GL}_n)$, $(\mathrm{Sp}_{2n}, \mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q})$
3. Group mfd: $(G, K) = (G_1 \times G_1, \mathrm{diag}\, G_1)$ G_1 : reductive grp

$P \subset G$: parabolic of G & $Q \subset K$: parabolic of K

Notation

$\mathfrak{X}_P := G/P$: partial flag var & $\mathcal{Z}_Q := K/Q$: pfv for K

$\mathfrak{X}_P \times \mathcal{Z}_Q$: double flag variety $\curvearrowleft K$ acts diagonally

Cf. Toshi's lecture $\cdots$ Property (PP') for minimal psg

Finiteness of orbits

$\mathfrak{X}_P := G/P$ & $\mathcal{Z}_Q := K/Q$: partial flag for G & K respectively

$\mathfrak{X}_P \times \mathcal{Z}_Q$: double flag variety (= DFV) $\curvearrowleft K$ acts diagonally

Definition 3.2

$$\#(\mathfrak{X}_P \times \mathcal{Z}_Q)/K < \infty \overset{\text{def}}{\Longleftrightarrow} \text{finite type}$$

Example 3.3 (Double flag var (DFV) of finite type [NO11])

Type AIII : $G/K = \mathrm{GL}_n/\mathrm{GL}_p \times \mathrm{GL}_q$ $(n = p + q)$

P	Q_1	Q_2	$\mathfrak{X}_P$	$\mathcal{Z}_Q$
any	mirabolic	GL_q	$\mathfrak{X}_P$	$\mathbb{P}(\mathbb{C}^p)$
any	GL_p	mirabolic	$\mathfrak{X}_P$	$\mathbb{P}(\mathbb{C}^q)$
$(\lambda_1, \lambda_2, \lambda_3)$	maximal	maximal	$\mathfrak{X}_P$	$\mathscr{G}r_k(\mathbb{C}^p) \times \mathscr{G}r_\ell(\mathbb{C}^q)$
$(\lambda_1, \lambda_2, 1)$	any	any	$\mathfrak{X}_P$	$\mathcal{Z}_Q$
maximal	any	any	$\mathscr{G}r_m(\mathbb{C}^n)$	$\mathcal{Z}_Q$

Classification of DFV of finite type

If $(G, K) = (G_1 \times G_1, \mathrm{diag}\, G_1)$, DFV becomes triple flag variety (TFV)

$$G/P \times K/Q = \Big((G_1 \times G_1)/(P_1 \times P_2)\Big) \times (G_1/P_3) = G_1/P_1 \times G_1/P_2 \times G_1/P_3$$

$\exists$ classification of TFV of finite type
$\cdots$ Magyar-Weymann-Zelevinsky [MWZ99] [MWZ00] (see also [Mat13])

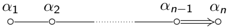
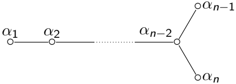
How about general DFV?

Theorem 3.4 ([HNOO13])

Let $B_G \subset G$: Borel in G & $B_K \subset K$: Borel in K

1. $\exists$ *Classification of $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ of finite type (Q is Borel)* $\iff$ *K-spherical G/P*
2. $\exists$ *Classification of $\mathfrak{X}_{B_G} \times \mathcal{Z}_Q$ of finite type (P is Borel)*

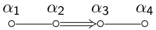
We will show the list of $\mathfrak{X}_P \times \mathcal{Z}_{B_K}$ of finite type, ... but very briefly (quoted from [HNOO13])

$\mathfrak{g}$	$\mathfrak{k}$	$\Pi \setminus J$ $(P = P_J)$
$\mathfrak{sl}_{n+1}$		
$\mathfrak{sl}_{n+1}$	$\mathfrak{so}_{n+1}$	$\{\alpha_i\}(\forall i)$
$\mathfrak{sl}_{2m}$ $2m = n+1$	$\mathfrak{sp}_m$	$\{\alpha_i\}(\forall i)$, $\{\alpha_i, \alpha_{i+1}\}(\forall i)$, $\{\alpha_1, \alpha_i\}(\forall i)$, $\{\alpha_i, \alpha_n\}(\forall i)$, $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$, $\{\alpha_1, \alpha_2, \alpha_n\}$, $\{\alpha_1, \alpha_{n-1}, \alpha_n\}$
$\mathfrak{sl}_{p+q}$ $p+q = n+1$	$\mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbb{C}$ $1 \leq p \leq q$	$\{\alpha_i\}(\forall i)$, $\{\alpha_i, \alpha_{i+1}\}(\forall i)$, $\{\alpha_1, \alpha_i\}(\forall i)$, $\{\alpha_i, \alpha_n\}(\forall i)$, $\{\alpha_i, \alpha_j\}(\forall i, j)$ if $p = 2$, any subset of Π if $p = 1$
$\mathfrak{so}_{2n+1}$		
$\mathfrak{so}_{p+q}$ $p+q = 2n+1$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$ $1 \leq p \leq q$	$\{\alpha_1\}$, $\{\alpha_n\}$, $\{\alpha_i\}(\forall i)$ if $p = 2$, any subset of Π if $p = 1$
$\mathfrak{so}_{2n}$		
$\mathfrak{so}_{p+q}$ $p+q = 2n$ $n \geq 4$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$ $1 \leq p \leq q$	$\{\alpha_1\}$, $\{\alpha_{n-1}\}$, $\{\alpha_n\}$, $\{\alpha_i\}(\forall i)$ if $p = 2$, $\{\alpha_i, \alpha_{n-1}\}(\forall i)$ if $p = 2$, $\{\alpha_i, \alpha_n\}(\forall i)$ if $p = 2$, any subset of Π if $p = 1$
$\mathfrak{so}_{2n}$ $n \geq 4$	$\mathfrak{sl}_n \oplus \mathbb{C}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_{n-1}\}$, $\{\alpha_n\}$, $\{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_{n-1}\}$, $\{\alpha_1, \alpha_n\}$, $\{\alpha_{n-1}, \alpha_n\}$, $\{\alpha_2, \alpha_3\}$ if $n = 4$

$\mathfrak{sp}_n$

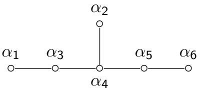
$\mathfrak{sp}_n$	$\mathfrak{sl}_n \oplus \mathbb{C}$	$\{\alpha_1\}$, $\{\alpha_n\}$
$\mathfrak{sp}_{p+q}$ $p+q=n$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$ $1 \le p \le q$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_n\}$, $\{\alpha_1, \alpha_2\}$, $\{\alpha_i\}(\forall i)$ if $p \le 2$, $\{\alpha_i, \alpha_j\}(\forall i, j)$ if $p = 1$

$\mathfrak{f}_4$



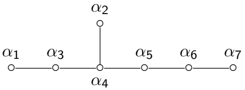
$\mathfrak{f}_4$	$\mathfrak{so}_9$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_4\}$, $\{\alpha_1, \alpha_4\}$

$\mathfrak{e}_6$



$\mathfrak{e}_6$	$\mathfrak{sp}_4$	$\{\alpha_1\}$, $\{\alpha_6\}$
$\mathfrak{e}_6$	$\mathfrak{sl}_6 \oplus \mathfrak{sl}_2$	$\{\alpha_1\}$, $\{\alpha_6\}$
$\mathfrak{e}_6$	$\mathfrak{so}_{10} \oplus \mathbb{C}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_5\}$, $\{\alpha_6\}$, $\{\alpha_1, \alpha_6\}$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_5\}$, $\{\alpha_6\}$, $\{\alpha_1, \alpha_2\}$, $\{\alpha_2, \alpha_6\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_5, \alpha_6\}$

$\mathfrak{e}_7$



$\mathfrak{e}_7$	$\mathfrak{sl}_8$	$\{\alpha_7\}$
$\mathfrak{e}_7$	$\mathfrak{so}_{12} \oplus \mathfrak{sl}_2$	$\{\alpha_7\}$
$\mathfrak{e}_7$	$\mathfrak{e}_6 \oplus \mathbb{C}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_7\}$

Moment maps

$X := \mathfrak{X}_P \times \mathcal{Z}_Q \curvearrowleft K$: diagonal K-action

$$\mathfrak{X}_P = \{\mathfrak{p}' \mid \mathfrak{p}' \overset{\mathrm{Ad}\,G}{\sim} \mathfrak{p}\} : \text{collection of psg conj to } \mathfrak{p}$$

$$T^*\mathfrak{X}_P = \{(\mathfrak{p}', \xi) \in \mathfrak{X}_P \times \mathfrak{g}^* \mid \xi \in (\mathfrak{p}')^{\perp}\} \simeq G \times_P \mathfrak{u}_P$$

Want to apply Steinberg theory to X/K:

$$\begin{array}{ccccc}
T^*X & = & T^*\mathfrak{X}_P \times T^*\mathcal{Z}_Q & \ni & \big((\mathfrak{p}', \xi), (\mathfrak{q}', \eta)\big) \\
 & & \downarrow \mu_{\mathfrak{X}_P} \times \mu_{\mathcal{Z}_Q} & & \downarrow \\
 & & \mathfrak{g}^* \times \mathfrak{k}^* & \ni & (\xi, \eta) \\
 & \searrow \mu_X & \downarrow \alpha & & \downarrow \\
 & & \mathfrak{k}^* & \ni & \xi|_{\mathfrak{k}} + \eta
\end{array}$$

$$\mu_{\mathfrak{X}_P}(T^*\mathfrak{X}_P) = G \cdot \mathfrak{u}_P = \overline{\mathscr{O}_P^G} \subset \mathcal{N}(\mathfrak{g}) : \text{Richardson orbit for } P$$

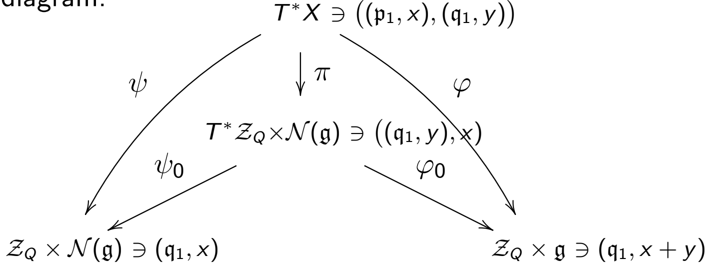
conormal variety for DFV $X = \mathfrak{X}_P \times \mathcal{Z}_Q$

$\mathcal{Y}_X := \mu_X^{-1}(0) = \bigcup_{\mathbb{O} \in X/K} \overline{T^*_{\mathbb{O}} X}$: conormal variety

Notation: $x^\theta := \frac{1}{2}(x + \theta(x)) \in \mathfrak{k}$

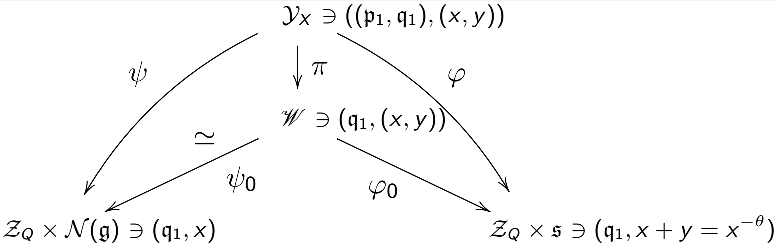
$$\begin{array}{ccc} \mathfrak{g} \ni x & \longleftrightarrow & \xi \in \mathfrak{g}^* \\ x^\theta & \longleftrightarrow & \xi\big|_{\mathfrak{k}} \end{array}$$

Consider diagram:



Put $\mathscr{W} := \pi(\mathcal{Y}_X) \subset T^*(\mathcal{Z}_Q) \times \mathcal{N}(\mathfrak{g})$: closed subvariety
Note : $((\mathfrak{p}_1, \mathfrak{q}_1), (x, y)) \in \mathcal{Y}_X \implies x + y = x - x^\theta =: x^{-\theta} \in \mathfrak{s}$

Exotic & enhanced nilpotent variety



Assumption 5.1

We assume $\mathscr{E}_X := \varphi(\mathcal{Y}_X) \subset \mathcal{Z}_Q \times \mathcal{N}(\mathfrak{s})$ in the following

Definition 5.2

1. $\mathscr{E}_X = \varphi(\mathcal{Y}_X) \subset \mathcal{Z}_Q \times \mathcal{N}(\mathfrak{s})$ is called exotic nilpotent variety
2. $\mathscr{R} := \psi(\mathcal{Y}_X) \subset \mathcal{Z}_Q \times \mathcal{N}(\mathfrak{g})$ is called enhanced Richardson variety

Related works

1. Robinson-Schensted correspondence for mirabolic triple flag variety
   ··· Finkelberg-Ginzburg-Travkin[FGT09], Travkin [Tra09]

2. Exotic nilpotent cone, Springer representations
   ··· Syu Kato [Kat09], [Kat11]

3. Enhanced nilpotent cone ··· Achar-Henderson [AH08]

4. Exotic character sheaves over $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$
   1. Exotic RS correspondence ··· Henderson-Trapa [HT12]
   2. Exotic character sheaves ··· Shoji-Sorlin [SS13], [SS14a], [SS14b]

5. Shoji's recent works on exotic symmetric spaces, enhanced variety, multiple Kostka polynomials, Springer correspondence for enhanced varieties of higher level ··· [Sho14], [SS15], [Sho15a], [Sho15b], [Sho15c]

Theorem 5.3 (Type AIII : $G/K = \mathrm{GL}_n/\mathrm{GL}_p \times \mathrm{GL}_q$ $(n = p + q)$)

1. *In the following cases, Assumption 5.1 holds, i.e.,*
   $\mathscr{E}_X \subset \mathcal{Z}_Q \times \mathcal{N}(\mathfrak{s})$ *(the exotic nilpotent variety is well-defined)*

P	Q_1	Q_2	$\#\mathscr{E}/K < \infty$
any	*mirabolic*	GL_q	*yes*
any	GL_p	*mirabolic*	*yes*
$(\lambda_1, \lambda_2, \lambda_3)$	*maximal*	*maximal*	???
$(\lambda_1, \lambda_2, 1)$	*maximal*	*any*	???
$(\lambda_1, \lambda_2, 1)$	*any*	*maximal*	???
maximal	*any*	*any*	*see below*

2. *If the pair* (P, Q) *is* (a) $($*any*, *mirabolic*$)$ *or* (b) *P and Q have abelian nilradicals, then* $\#\mathscr{E}_X/K < \infty$ *holds*

Remark 5.4

For a large class of $X = \mathfrak{X}_P \times \mathcal{Z}_Q$ of finite type, it seems likely to hold

$$\mathscr{E}_X \subset \mathcal{Z}_Q \times \mathcal{N}(\mathfrak{s}) \quad \text{and} \quad \#\mathscr{E}_X/K < \infty \qquad \text{(but not always)}$$

Orbit correspondence

Assume that
1. DFV $X = \mathfrak{X}_P \times \mathcal{Z}_Q$ is of finite type: $\# X/K < \infty$
2. Exotic nullcone $\mathscr{E}_X \subset \mathcal{Z}_Q \times \mathcal{N}(\mathfrak{s})$ is well-defined
3. finiteness of exotic nilpotent K-orbits: $\# \mathscr{E}_X / K < \infty$

$\varphi : \mathcal{Y}_X \to \mathscr{E}_X$: K-equiv map
$\rightsquigarrow$ For $\forall \mathbb{O} \in X/K$, $\exists \mathscr{O} \in \mathscr{E}_X/K$ s.t. $\varphi(\overline{T^*_{\mathbb{O}} X}) = \overline{\mathscr{O}}$

This induces a map

$$\Phi : X/K \simeq \mathrm{Irr}(\mathcal{Y}_X) \ni \overline{T^*_{\mathbb{O}} X} \mapsto \mathscr{O} \in \mathscr{E}_X/K$$

Definition 6.1

The correspondence

$$\text{Orbits on } X \longleftrightarrow \text{Orbits on } \mathscr{E}_X$$

is called Exotic Robinson-Schensted correspondence

Though the terminology is somewhat different from the original meaning...

Exotic Grassmannian: setting

2n-dim vector space $V := \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n =: V_1 \oplus V_2$

$(G, K) = (\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$ Hermitian symmetric pair of type AIII

$$G = \mathrm{GL}(V), K = K_1 \times K_2 = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$$

1. $P = \mathrm{Stab}_G(V_1)$: maximal parabolic stabilizing V_1

$$= P_{(n,n)} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in \mathrm{GL}_n, b \in \mathrm{M}_n \right\}$$

$$\rightsquigarrow \; G/P = \mathscr{G}r_n(V) \; : \; \text{Grassmannian of } n\text{-spaces in } V$$

2. $Q = Q_1 \times \mathrm{GL}_n = \mathrm{Stab}_K(L_0)$: mirabolic subgrp (stabilizer of a line)

$$Q_1 = P_{(1,n-1)} \subset \mathrm{GL}_n = K_1 \; : \; \text{mirabolic in } K_1$$

$$\rightsquigarrow \; K/Q = \mathbb{P}(V_1) \; : \; \text{projective space (collection of lines)}$$

Exercise on Exotic Grassmannian

$$X = \mathfrak{X}_P \times \mathcal{Z}_Q = \mathrm{GL}_{2n}/P_{(n,n)} \times \mathrm{GL}_n/P_{(1,n-1)} \quad : \text{DFV}$$

$$\simeq \mathscr{G}r_n(V) \times \mathbb{P}(V_1) \quad \text{with } \mathrm{GL}_n \times \mathrm{GL}_n\text{-action}$$

Warming up exercise:

What are K-orbits in $\mathscr{G}r_n(V)$ through $U \subset V$ (n-dim subspace)?

Answer:

Dimensions $(r, s) = (\dim U \cap V_1, \dim U \cap V_2)$ classify K-orbits

Coding by Young tableaux; explaining by example $n = 3$

0 0	0 0	2 0	0 2	0 0	1 0	0 1	0 1	1 0	1 1
0 2	2 0	2 0	0 2	1 1	2 0	0 2	1 1	1 1	1 1
2 2	2 2	2 0	0 2	2 2	2 1	1 2	1 2	2 1	1 1

Coding K-orbits on $\mathscr{G}r_n(V)$ by Young tableaux

Example $n = 3$

$$
\begin{array}{|c|c|}\hline 0&0\\\hline 0&2\\\hline 2&2\\\hline\end{array}\;
\begin{array}{|c|c|}\hline 0&0\\\hline 2&0\\\hline 2&2\\\hline\end{array}\;
\begin{array}{|c|c|}\hline 2&0\\\hline 2&0\\\hline 2&0\\\hline\end{array}\;
\begin{array}{|c|c|}\hline 0&2\\\hline 0&2\\\hline 0&2\\\hline\end{array}\;
\begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline 2&2\\\hline\end{array}\quad
\begin{array}{|c|c|}\hline 1&0\\\hline 2&0\\\hline 2&1\\\hline\end{array}\;
\begin{array}{|c|c|}\hline 0&1\\\hline 0&2\\\hline 1&2\\\hline\end{array}\;
\begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline 1&2\\\hline\end{array}\;
\begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline 2&1\\\hline\end{array}\;
\begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline 1&1\\\hline\end{array}
$$

T : 2-column Young tableaux filled with $0, 1, 2$

Rules: T_i : i-th column of the tableau T ($i = 1, 2$)

We fill T with $0, 1, 2$ in increasing order from top to bottom subject to

- T_1 contains 2 for r-times, T_2 contains 2 for s-times
- 1 appears in T_1 and T_2 in the same number
- The sum of figures in T is $2n$

Coding K-orbits in Exotic Grassmannian $\mathscr{G}r_n(V) \times \mathbb{P}(V_1)$

Marked tableaux : $\quad (\text{tableau}, \text{mark}) = (T, i)$

$U \subset V$: n-dim subspace $\quad \& \quad L \subset V_1$: a line

- K-orbits in $\mathscr{G}r_n(V)$ through $U \leftrightarrow T$
- $L \leftrightarrow$ marking on T_1 (a choice of $0, 1, 2$ from T_1)

Example 6.2

$n = 3$

$$(\begin{array}{|c|c|}\hline 0&0\\\hline 0&2\\\hline 2&2\\\hline\end{array}, 0)\ (\begin{array}{|c|c|}\hline 0&0\\\hline 0&2\\\hline 2&2\\\hline\end{array}, 2)\ (\begin{array}{|c|c|}\hline 0&0\\\hline 2&0\\\hline 2&2\\\hline\end{array}, 0)\ (\begin{array}{|c|c|}\hline 0&0\\\hline 2&0\\\hline 2&2\\\hline\end{array}, 2)\ (\begin{array}{|c|c|}\hline 2&0\\\hline 2&0\\\hline 2&0\\\hline\end{array}, 2)\ (\begin{array}{|c|c|}\hline 0&2\\\hline 0&2\\\hline 0&2\\\hline\end{array}, 0)$$

$$(\begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline 2&2\\\hline\end{array}, 0)\ (\begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline 2&2\\\hline\end{array}, 1)\ (\begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline 2&2\\\hline\end{array}, 2)\ (\begin{array}{|c|c|}\hline 1&0\\\hline 2&0\\\hline 2&1\\\hline\end{array}, 1)\ (\begin{array}{|c|c|}\hline 1&0\\\hline 2&0\\\hline 2&1\\\hline\end{array}, 2)\ (\begin{array}{|c|c|}\hline 0&1\\\hline 0&2\\\hline 1&2\\\hline\end{array}, 0)$$

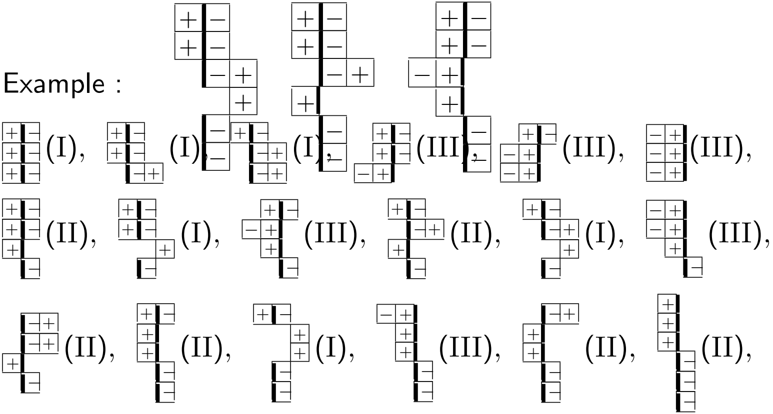
$$(\begin{array}{|c|c|}\hline 0&1\\\hline 0&2\\\hline 1&2\\\hline\end{array}, 1)\ (\begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline 1&2\\\hline\end{array}, 0)\ (\begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline 1&2\\\hline\end{array}, 1)\ (\begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline 2&1\\\hline\end{array}, 1)\ (\begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline 2&1\\\hline\end{array}, 2)\ (\begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline 1&1\\\hline\end{array}, 1)$$

Exotic nilpotent orbits

striped signed Young diagram:

classifying K-orbits in $\mathbb{P}(V_1) \times \mathcal{N}(\mathfrak{s})$ (Johnson[Joh10])

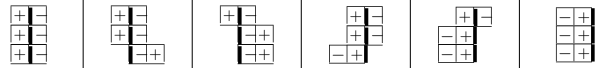
Example :



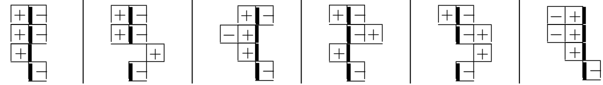
- type (I): $\boxed{+}\!\!\mid\!\boxed{-}$ appears & the rest are to the right of the mid-bar
- type (II): $\boxed{+}\!\mid$ appears to the left of the mid-bar

Correspondence: $(G, K) = (\mathrm{GL}_6, \mathrm{GL}_3 \times \mathrm{GL}_3), X = \mathscr{G}r_3(\mathbb{C}^6) \times \mathbb{P}(\mathbb{C}^3)$

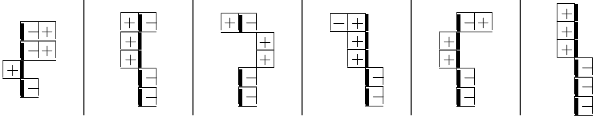
$\left(\begin{array}{\|c\|c\|}\hline 2&0\\\hline 2&0\\\hline 2&0\\\hline\end{array}, 2\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&0\\\hline 2&0\\\hline 2&2\\\hline\end{array}, 2\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&0\\\hline 0&2\\\hline 2&2\\\hline\end{array}, 2\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&0\\\hline 2&0\\\hline 2&2\\\hline\end{array}, 0\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&0\\\hline 0&2\\\hline 2&2\\\hline\end{array}, 0\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&2\\\hline 0&2\\\hline 0&2\\\hline\end{array}, 0\right)$



$\left(\begin{array}{\|c\|c\|}\hline 1&0\\\hline 2&0\\\hline 2&1\\\hline\end{array}, 1\right)$	$\left(\begin{array}{\|c\|c\|}\hline 1&0\\\hline 2&0\\\hline 2&1\\\hline\end{array}, 2\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&0\\\hline 1&1\\\hline 2&2\\\hline\end{array}, 0\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&0\\\hline 1&1\\\hline 2&2\\\hline\end{array}, 1\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&0\\\hline 1&1\\\hline 2&2\\\hline\end{array}, 2\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&1\\\hline 0&2\\\hline 1&2\\\hline\end{array}, 0\right)$



$\left(\begin{array}{\|c\|c\|}\hline 0&1\\\hline 0&2\\\hline 1&2\\\hline\end{array}, 1\right)$	$\left(\begin{array}{\|c\|c\|}\hline 1&0\\\hline 1&1\\\hline 2&1\\\hline\end{array}, 1\right)$	$\left(\begin{array}{\|c\|c\|}\hline 1&0\\\hline 1&1\\\hline 2&1\\\hline\end{array}, 2\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&1\\\hline 1&1\\\hline 1&2\\\hline\end{array}, 0\right)$	$\left(\begin{array}{\|c\|c\|}\hline 0&1\\\hline 1&1\\\hline 1&2\\\hline\end{array}, 1\right)$	$\left(\begin{array}{\|c\|c\|}\hline 1&1\\\hline 1&1\\\hline 1&1\\\hline\end{array}, 1\right)$



Exotic Robinson-Schensted correspondence

Recall the setting:
$V = \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n = V_1 \oplus V_2$

Consider $(G, K) = (\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$ Hermitian symm pair of type AIII

$X = G/P \times K/Q = \mathscr{G}r_n(V) \times \mathbb{P}(V_1)$: DFV on which K acts

Exotic nullcone : $\mathscr{E}_X = \mathbb{P}(V_1) \times \mathcal{N}_2(\mathfrak{s})$: marked 2-step nilpotents

Theorem 7.1 (Fresse-N)

∃ 1-to-1 correspondence

$$X/K \ni \mathbb{O} \mapsto \mathcal{O} \in \mathscr{E}_X/K$$

via moment map

$$\varphi(\overline{T^*_{\mathbb{O}}X}) = \overline{\mathcal{O}}$$

Remark 7.2

For 2-step nilpotents, the Spaltenstein variety is irreducible

General orbit correspondence

General situation where $V = \mathbb{C}^N = \mathbb{C}^p \oplus \mathbb{C}^q = V_1 \oplus V_2 \quad (N = p + q)$

$$(G, K) = (\mathrm{GL}_N, \mathrm{GL}_p \times \mathrm{GL}_q)$$

$X = G/P \times K/Q = \mathscr{G}r_k(V) \times \mathbb{P}(V_1)$: DFV on which K acts

Exotic nullcone : $\mathscr{E}_X \subset \mathbb{P}(V_1) \times \mathcal{N}_2(\mathfrak{s})$: marked 2-step nilpotents

Theorem 7.3 (Fresse-N)

$\exists$ *finite-to-1 map (at most 2-to-1)*

$$X/K \ni \mathbb{O} \mapsto \mathcal{O} \in \mathscr{E}_X/K$$

via moment map

$$\varphi(\overline{T^*_{\mathbb{O}} X}) = \overline{\mathcal{O}}$$

We can describe the map concretely.

Can extend this to type CI where $(G, K) = (\mathrm{Sp}_{2n}, \mathrm{GL}_n)$ & expecting more.
Let us give a simplest example...

Example: Further correspondence

$(G, K) = (\mathrm{GL}_4, \mathrm{GL}_3 \times \mathrm{GL}_1)$

K acts on $X = \mathscr{G}r_k(\mathbb{C}^4) \times \mathbb{P}(\mathbb{C}^3)$ $(k = 1, 2)$

Orbit corresp. map $\phi : \Theta_2^k = X/K \longrightarrow \Pi_2^k = \mathscr{E}/K$

$\phi^{-1}(T)$ for $k = 1$	$\left(\begin{array}{ll} 0 & 0 \\ 0 & \\ 2 & \end{array}, 2\right)$	$\left(\begin{array}{ll} 0 & 0 \\ 0 & \\ 2 & \end{array}, 0\right)$	$\left(\begin{array}{ll} 0 & 1 \\ 0 & \\ 1 & \end{array}, 1\right)$	$\left(\begin{array}{ll} 0 & 2 \\ 0 & \\ 0 & \end{array}, 0\right)$	$\left(\begin{array}{ll} 0 & 1 \\ 0 & \\ 1 & \end{array}, 0\right)$
$\phi^{-1}(T)$ for $k = 2$	$\left(\begin{array}{ll} 0 & 1 \\ 1 & \\ 2 & \end{array}, 2\right)$	$\left(\begin{array}{ll} 0 & 0 \\ 2 & \\ 2 & \end{array}, 2\right), \left(\begin{array}{ll} 0 & 0 \\ 2 & \\ 2 & \end{array}, 0\right)$	$\left(\begin{array}{ll} 0 & 2 \\ 0 & \\ 2 & \end{array}, 2\right), \left(\begin{array}{ll} 0 & 1 \\ 1 & \\ 2 & \end{array}, 1\right)$	$\left(\begin{array}{ll} 0 & 2 \\ 0 & \\ 2 & \end{array}, 0\right)$	$\left(\begin{array}{ll} 0 & 1 \\ 1 & \\ 2 & \end{array}, 0\right)$
$T \in \Pi_2^k$	$\begin{array}{cccc} + & \mathbf{\vert} & - & \\ & & & + \\ & & & + \end{array}$	$\begin{array}{ccc} + & \mathbf{\vert} & - \\ + & \mathbf{\vert} & \\ + & \mathbf{\vert} & \end{array}$	$\begin{array}{cccc} & \mathbf{\vert} & - & + \\ + & \mathbf{\vert} & & \\ + & \mathbf{\vert} & & \end{array}$	$\begin{array}{ccc} - & + & \mathbf{\vert} \\ & + & \mathbf{\vert} \\ & + & \mathbf{\vert} \end{array}$	$\begin{array}{ccc} + & \mathbf{\vert} & \\ + & \mathbf{\vert} & \\ + & \mathbf{\vert} & \\ & \mathbf{\vert} & - \end{array}$

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