Non-Siegel Eisenstein series for symplectic groups

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Notation

- $n \in \mathbb{Z}_{\geq 1}$; J_n $n \times n$ matrix with $1'$s on the opposite diagonal, and zeros everywhere else.
- $$Sp_{2n}(F) = \left\{ g \in GL_{2n}(F) : g^t \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} g = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \right\},$$
  $F \in \{\mathbb{Q}, \mathbb{Q}_p, \mathbb{R}, \mathbb{A}\}$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$.
- $K_p = Sp_{2n}(\mathbb{Z}_p)$ for $p < \infty$, K_∞ be the fixed point set of a Cartan involution (e.g., transposed inverse) on $Sp_{2n}(\mathbb{R})$, $K_\infty \cong U(n)$. $K = \prod_{p \leq \infty} K_p$.

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Borel subgroup; roots

- B=the upper triangular matrices in Sp_{2n}; a Borel subgroup which we fix. $\rightsquigarrow$ the standard parabolic subgroups.
- $$T(F) = \{diag(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots t_2^{-1}, t_1^{-1}); t_1, \ldots, t_n \in F^*\}.$$
- $W = W(B, T)$ be the Weyl group of Sp_{2n}; $W \cong S_n \ltimes \mathbb{Z}_2^n$, . The Weyl group elements corresponding to S_n–permutations, and to $\mathbb{Z}_n$–the sign changes. The action of $p \in S_n$ is given by

  $p(diag(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots t_2^{-1}, t_1^{-1})) = (t_{p^{-1}(1)}, t_{p^{-1}(2)}, \ldots, t_{p^{-1}(n)}, \ldots)$

  and the action of $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{Z}_2^n$ is given by

  $$\varepsilon(diag(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots t_2^{-1}, t_1^{-1})) = diag(t_1^{\varepsilon_1}, t_2^{\varepsilon_2}, \ldots, t_n^{\varepsilon_n}, \ldots).$$
- for $w \in W$ and $t \in T$, $(w\phi)(t) = \phi(w^{-1}t)$. In more words, if $w = p\varepsilon$ and $\phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n$ we have

  $$p\varepsilon(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n) = \phi_{p^{-1}(1)}^{\varepsilon_{p^{-1}(1)}} \otimes \cdots \otimes \phi_{p^{-1}(n)}^{\varepsilon_{p^{-1}(n)}} \quad (1)$$

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- $$\alpha_i(diag(t_1, t_2, \dots, t_n, t_n^{-1}, \dots t_2^{-1}, t_1^{-1}) = t_i t_{i+1}^{-1}, \ i = 1, 2, \dots n-1,$$

  $$\alpha_n(diag(t_1, t_2, \dots, t_n, t_n^{-1}, \dots t_2^{-1}, t_1^{-1}) = t_n^2$$

  are the simple roots (Δ) corresponding to our choice of Borel. We also use $e_i - e_{i+1}$ to denote $\alpha_i, \ i = 1, \dots, n-1$ and $2e_n$ to denote α_n. In the same way, we can describe the set of all positive roots (with respect to B) as $\Sigma^+ = \{e_i - e_j, 1 \leq i < j \leq n\} \cup \{e_i + e_j, 1 \leq i < j \leq n\} \cup \{2e_i : 1 \leq i \leq n\}$. For $\alpha_i \in \Delta$ we denote

  $$W^{\alpha_i} = \{w = p\varepsilon \in W : w\alpha_i > 0\}.$$

- $\Omega \subset \Delta \rightsquigarrow [W/W_\Omega] = \cap_{\alpha \in \Omega} W^\alpha$ (this is indeed a set of representatives of left cosets of W modulo its subgroup W_Ω).

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Normalization factors for local intertwining operators

$$r(\Lambda_{s,p}, w) = \prod_{\alpha\in\Sigma^+, w(\alpha)<0} \frac{L(1, \Lambda_{s,p} \circ \check{\alpha})\epsilon(1, \Lambda_{s,p} \circ \check{\alpha}, \psi_p)}{L(0, \Lambda_{s,p} \circ \check{\alpha})},$$

i.e., globally

$$r(\Lambda_s, w)^{-1} = \prod_{\alpha\in\Sigma^+, w(\alpha)<0} \frac{L(0, \Lambda_s \circ \check{\alpha})}{L(1, \Lambda_s \circ \check{\alpha})\epsilon(1, \Lambda_s \circ \check{\alpha})}. \tag{2}$$

Description of the Weyl group elements

Lemma

Let $1 \leq i \leq n$ and let $0 \leq j \leq i$. Denote by Y_j^i the set of all $p\varepsilon \in W$ such that the following six conditions are satisfied:

1. $\varepsilon_k = 1$, for $1 \leq k \leq j$;
2. $p(k_1) < p(k_2)$, for $1 \leq k_1 < k_2 \leq j$;
3. $\varepsilon_k = -1$, for $j + 1 \leq k \leq i$;
4. $p(k_1) > p(k_2)$ for $j + 1 \leq k \leq i$;
5. $\varepsilon_k = 1$, for $i + 1 \leq k \leq n$;
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Then $[W/W_{\Delta \setminus \{\alpha_i\}}] = \cup_{0 \leq j \leq i} Y_j^i$.

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Lemma

Assume $\chi^2 = 1$. *Assume* $w' = p'\varepsilon' \in [w]_\chi$ *with* $w = p\varepsilon \in Y^i_{j_1}$ *and* $w' \in Y^i_{j_1}$. *Then* $j_1 = j_2$ *or* $j_2 = i - 2s - j_1$.

Lemma

Assume that $\chi^2 = 1, \chi \neq 1$ *and* $2s \in \mathbb{Z}$ *with* $0 < 2s \leq i - 1$*. Let* $w_1 = p_1\varepsilon_1,\ w_2 = p_2\varepsilon_2 \in [W/W_{\Delta\setminus\{\alpha_i\}}], w_1 \neq w_2$ *and* $w_1 \in [w_2]_\chi$*. Then, one of the following holds*

- $w_1, w_2 \in Y^i_j$ *for some* $0 < j < i$*. Then, we have* $p_1(k) = p_2(k), k = i+1, \ldots, n$*. For* $k = 1, \ldots, j$ *we have* $p_1(k) = p_2(k)$ *or* $p_1(k) = p_2(i + 1 - 2s - k)$*. In the latter case, we must have* $k \leq \min\{i - 2s - j, j\}$*. For* $k = j + 1, \ldots, i$ *we have* $p_1(k) = p_2(k)$ *or* $p_1(k) = p_2(i + 1 - 2s - k)$*. In the latter case, we must have* $k \geq \max\{i + 1 - 2s - j, j + 1\}$*.*

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- If $w_1 = w_0$ (the longest element), then $w_2 = p\varepsilon$, where $w_2 \in Y^i_{i-2s}$ and $p(k) = 2s + k,\ k = 1, 2, \ldots, i - 2s$ $p(k) = i + 1 - k,\ k = i - 2s + 1, \ldots, i$ and $p(k) = k,\ k = i + 1, \ldots, n$. Note that Y^i_0 does not consist of w_0 alone.

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Remark

1. If $\chi \neq 1$ and $j \geq i - 2s + 1$, then $[w]_\chi = \{w\}$.
2. If $\chi = 1$ and $w' = p'\varepsilon \in [w = p\varepsilon]_1$ with $2s > i$ (so that $w, w' \in Y_j^i$ for some j) then $p|_{\{1,2,\ldots,j\}} = p'|_{\{1,2,\ldots,j\}}$.
3. With the assumptions of Lemma 3 with $w_1, w_2 \in Y_j^i$, if, for $k \in \{1, 2, \ldots, j\}$ $p_1(k) = p_2(i + 1 - 2s - k)$, then $p_1(i + 1 - 2s - k) = p_2(k)$. All such k's belong to *intervals of change-i.e. the shortest intervals on which these interchanges can take place. Then we can describe whole $[w]_\chi \cap Y_j^i$ by prescribing whether some of these intervals are included or not; similarily as in the case of GL_n or the Siegel case of Sp_{2n}. These cases can be modified to our, more complicated case (the global normalization factors are much more complicated in the non-Siegel case)*

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1. If $\chi \neq 1$ and $j \geq i - 2s + 1$, then $[w]_\chi = \{w\}$.
2. If $\chi = 1$ and $w' = p'\varepsilon \in [w = p\varepsilon]_1$ with $2s > i$ (so that $w, w' \in Y_j^i$ for some j) then $p|_{\{1,2,\ldots,j\}} = p'|_{\{1,2,\ldots,j\}}$.
3. With the assumptions of Lemma 3 with $w_1, w_2 \in Y_j^i$, if, for $k \in \{1, 2, \ldots, j\}$ $p_1(k) = p_2(i + 1 - 2s - k)$, then $p_1(i + 1 - 2s - k) = p_2(k)$. All such k's belong to *intervals of change-i.e. the shortest intervals on which these interchanges can take place. Then we can describe whole* $[w]_\chi \cap Y_j^i$ *by prescribing whether some of these intervals are included or not; similarly as in the case of* GL_n *or the Siegel case of* Sp_{2n}. *These cases can be modified to our, more complicated case (the global normalization factors are much more complicated in the non-Siegel case)*

In the case of $\chi = 1$, these intervals of changes between $\{1, \ldots, j\}$ and $\{j + 1, \ldots, i\}$ are not enough to describe whole $[w]_1$; the orbits for $\chi = 1$ are much larger than orbits for $\chi \neq 1$ and their description and combinatorics are much more involved and the normalization factors have poles of even higher order.

Global normalization factors–continued

(2) for $w \in Y^i_{j_1}$ becomes a product of the following expressions:

$$\prod_{k=j_1+1}^{i} \frac{L(s-\frac{i-1}{2}+k-1,\chi)}{L(s-\frac{i-1}{2}+k,\chi)\epsilon(s-\frac{i-1}{2}+k,\chi)} \tag{3}$$

$$\prod_{k=j_1+1}^{i} \prod_{l=i+1}^{n} \frac{L(s-\frac{i-1}{2}+k+n-l,\chi)}{L(s-\frac{i-1}{2}+k+n-l+1,\chi)\epsilon(s-\frac{i-1}{2}+k+n-l+1,\chi)} \tag{4}$$

$$\prod_{\substack{k=1,\dots,j_1\\ l=i+1,\dots,n\\ p(k)>p(l)}} \frac{L(s-\frac{i-1}{2}+k+n-l,\chi)}{L(s-\frac{i-1}{2}+k+n-l+1,\chi)\epsilon(s-\frac{i-1}{2}+k+n-l+1,\chi)} \tag{5}$$

$$\prod_{\substack{k=1,\dots,j_1\\ l=j_1+1,\dots,i\\ p(k)>p(l)}} \frac{L(2s+k+l-i-1,\chi^2)}{L(2s+k+l-i,\chi^2)\epsilon(2s+k+l-i,\chi^2)} \tag{6}$$

$$\prod_{\substack{k=j_1+1,\dots,i\\ l=i+1,\dots,n\\ p(l)>p(k)}} \frac{L(s-\frac{i-1}{2}+k+l-n-2,\chi)}{L(s-\frac{i-1}{2}+k+l-n-1,\chi)\epsilon(s-\frac{i-1}{2}+k+l-n-1,\chi)} \tag{7}$$

$$\prod_{\substack{j_1+1\leq k,l\leq i\\ k<l}} \frac{L(2s-i+k+l-1,\chi^2)}{L(2s-i+k+l,\chi^2)\epsilon(2s-i+k+l,\chi^2)}. \tag{8}$$

- (6) is non-trivial for $w = p\varepsilon$ such that $p(j_1) > p(i)$.
- k_p the smallest $k \in \{1, 2, \ldots, j_1\}$ such that $p(k) > p(i)$.
- For $k \in \{k_p, k_p + 1, \ldots, j_1\}$ let l_k be the smallest element in $\{j_1 + 1, \ldots, i\}$ such that $p(k) > p(l_k)$ The expression (6) becomes

$$\prod_{k=k_p}^{j_1} \prod_{l=l_k}^{i} \frac{L(2s+k+l-i-1, \chi^2)}{L(2s+k+l-i, \chi^2)\epsilon(2s+k+l-i, \chi^2)} = \tag{9}$$

$$\prod_{k=k_p}^{j_1} \frac{L(2s+k-i-1+l_k, \chi^2)}{L(2s+k, \chi^2)}.$$

- (6) is non-trivial for $w = p\varepsilon$ such that $p(j_1) > p(i)$.
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$$\prod_{k=k_p}^{j_1} \prod_{l=l_k}^{i} \frac{L(2s + k + l - i - 1, \chi^2)}{L(2s + k + l - i, \chi^2)\epsilon(2s + k + l - i, \chi^2)} = \tag{9}$$

$$\prod_{k=k_p}^{j_1} \frac{L(2s + k - i - 1 + l_k, \chi^2)}{L(2s + k, \chi^2)}.$$

- (6) is non-trivial for $w = p\varepsilon$ such that $p(j_1) > p(i)$.
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$$\prod_{k=k_p}^{j_1} \frac{L(2s + k - i - 1 + l_k, \chi^2)}{L(2s + k, \chi^2)}.$$

- t_p an element of $\{i+1,\ldots,n\}$ such that $p(t_p) < p(j_1)$ and $p(t_p+1) > p(j_1)$
- for $l \in \{i+1,\ldots,t_p\}$ let m_l an element of $\{1,2,\ldots,j_1\}$ such that $p(l) < p(m_l)$ and $p(l) > p(m_l - 1)$. Then, (5) becomes

$$\prod_{l=i+1}^{t_p} \prod_{k=m_l}^{j_1} \frac{L(s-\frac{i-1}{2}+k+n-l,\chi)}{L(s-\frac{i-1}{2}+k+n-l+1,\chi)\epsilon(s-\frac{i-1}{2}+k+n-l+1,\chi)} \tag{10}$$

- t_p an element of $\{i+1, \ldots, n\}$ such that $p(t_p) < p(j_1)$ and $p(t_p+1) > p(j_1)$
- for $l \in \{i+1, \ldots, t_p\}$ let m_l an element of $\{1, 2, \ldots, j_1\}$ such that $p(l) < p(m_l)$ and $p(l) > p(m_l - 1)$. Then, (5) becomes

$$\prod_{l=i+1}^{t_p} \prod_{k=m_l}^{j_1} \frac{L(s - \frac{i-1}{2} + k + n - l, \chi)}{L(s - \frac{i-1}{2} + k + n - l + 1, \chi)\epsilon(s - \frac{i-1}{2} + k + n - l + 1, \chi)} \tag{10}$$

- $s_p \in \{i+1,\ldots,n\}$ such that $p(s_p) > p(i),\ p(s_p-1) < p(i)$.
- for $l \in \{s_p,\ldots,n\}$ we denote by r_l an element of $\{j_1+1,\ldots,i\}$ such that $p(r_l) < p(l)$ and $p(r_l-1) > p(l)$ and (7) becomes

$$\prod_{l=s_p}^{n}\prod_{k=r_l}^{i}\frac{L(s-\frac{i-1}{2}+k+l-n-2,\chi)}{L(s-\frac{i-1}{2}+k+l-n-1,\chi)\epsilon(s-\frac{i-1}{2}+k+l-n-1,\chi)}. \qquad (11)$$

- $s_p \in \{i+1,\ldots,n\}$ such that $p(s_p) > p(i),\ p(s_p-1) < p(i)$.
- for $l \in \{s_p,\ldots,n\}$ we denote by r_l an element of $\{j_1+1,\ldots,i\}$ such that $p(r_l) < p(l)$ and $p(r_l-1) > p(l)$ and (7) becomes

$$\prod_{l=s_p}^{n}\prod_{k=r_l}^{i}\frac{L(s-\frac{i-1}{2}+k+l-n-2,\chi)}{L(s-\frac{i-1}{2}+k+l-n-1,\chi)\epsilon(s-\frac{i-1}{2}+k+l-n-1,\chi)}. \tag{11}$$

The Langlands parameters of $\widehat{\sigma_1}$ and $\widehat{\sigma_2}$-the Siegel case

Theorem

Let $\frac{n-1}{2} - s \in \mathbb{Z}_{>0}$. *If* $\chi = 1$ *then*

$$\widehat{\sigma_1} = L(\nu^{\frac{n-1}{2}+s}, \nu^{\frac{n-3}{2}+s}, \dots, \nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \dots, \nu^1, \nu^1; \nu^0 \rtimes 1),$$

and

$$\widehat{\sigma_2} = L(\nu^{\frac{n-1}{2}+s}, \nu^{\frac{n-3}{2}+s}, \dots, \nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \dots, \nu^2, \nu^2, \nu^1; T_2'').$$

Here T_2'' *is the unique common (tempered) subquotient of* $\nu^0 \rtimes St_{SL_2(F)}$ *and* $\zeta(\nu^0, \nu^1) \rtimes 1$.

Theorem

If $\chi \neq 1$ we get analogously

$$\widehat{\sigma}_i = L(\nu^{\frac{n-1}{2}+s}, \nu^{\frac{n-3}{2}+s}, \ldots, \nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \ldots, \nu^{1}, \nu^{1}; T_i^0), \ i = 1, 2,$$

where $\chi \rtimes 1 = T_1^0 \oplus T_2^0$ (tempered representations of $SL_2(F)$).

The description of the image–the Siegel case

Theorem

Assume $\chi^2 = 1$ *with* $\chi_\infty = 1$*, and* $n \geq 3$ *with* $\frac{n-1}{2} - s \in \mathbb{Z}_{\geq 0}$*. Let* $\widehat{\sigma_{i,p}}, i = 1, 2$ *be the representations described above at the place* $p < \infty$*. Then, the Eisenstein series (0) has a pole of order one on* $I(s)$*.*

Description of image-the Siegel case–continuation

- Assume $0 < s < \frac{n-1}{2}$, $f = \otimes_{p \leq \infty} f_p \in I(s)$ and let S be a finite set of finite places, and for $p \notin S$, let f_p be the normalized spherical vector. For $S_1 \subset S$, we pick $f_p \in \widehat{\sigma_{1,p}}$ and for $p \in S_2 := S \setminus S_1$, we take $f_p \in \widehat{\sigma_{2,p}}$. Then, for such f, (0') is holomorphic if $|S_2|$ is odd, and if $|S_2|$ is even it has a pole of order one. In the latter case ((0')) gives an automorphic realization (in the space of automorphic forms $\mathcal{A}(Sp_{2n}(\mathbb{Q}) \setminus Sp_{2n}(\mathbb{A}))$ of a global irreducible representation having a local representation $\widehat{\sigma_{2,p}}$ on the places from S_2 and $\widehat{\sigma_{1,p}}$ as a local component elsewhere on finite places ($\widehat{\sigma_{1,p}}$ is spherical for $p \notin S$, $p < \infty$).