L-groups for double covers of Chevalley-Steinberg groups

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Outline

Covering groups after Brylinski and Deligne

The dual group

The L-group

Evidences and questions

Covering groups after Brylinski and Deligne

What is a covering group?

Let F be a field, and let $\mathbf{G}$ be a reductive group over F.

Definition (My working definition)

A cover of $\mathbf{G}$ over F is a pair $\tilde{\mathbf{G}} = (\mathbf{G}', n)$, where

- $\mathbf{K}_2 \hookrightarrow \mathbf{G}' \twoheadrightarrow \mathbf{G}$ is a central extension of $\mathbf{G}$ by $\mathbf{K}_2$;
- $1 \leq n$ (the degree) is such that $\#\mu_n(F) = n$.

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A "central extension of $\mathbf{G}$ by $\mathbf{K}_2$" was defined by Brylinski and Deligne (Pub. Math. IHES 94 (2001)). They classified such central extensions by root-theoretic data.

What does a covering group over $\mathbb{R}$ give us?

Let $\tilde{\mathbf{G}}$ be a degree 2 cover of a reductive group $\mathbf{G}$ over $\mathbb{R}$. Taking $\mathbb{R}$-points and applying the Hilbert symbol yields a topological central extension,

$$\mu_2 \hookrightarrow \tilde{G} \twoheadrightarrow G,$$

where $G = \mathbf{G}(\mathbb{R})$ and $\mu_2 = \{\pm 1\}$.

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Two questions:

1. What topological central extensions arise?
2. Why should one work with the Brylinksi-Deligne class of covers anyways?

Compact tori

Proposition

Let $\mathbf{S}$ *be an algebraic torus over* $\mathbb{R}$ *such that* $S = \mathbf{S}(\mathbb{R})$ *is compact. Then* every *topological central extension,*

$$\mu_2 \hookrightarrow \tilde{S} \twoheadrightarrow S$$

arises from a cover $\tilde{\mathbf{S}} = (\mathbf{S}', 2)$*.*

Proposition

Let $\mathbf{S}$ be an algebraic torus over $\mathbb{R}$ such that $S = \mathbf{S}(\mathbb{R})$ is compact. Then every *topological central extension,*

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arises from a cover $\tilde{\mathbf{S}} = (\mathbf{S}', 2)$.

Proof: A straightforward consequence of Brylinski-Deligne, §12.6.

N.B. the cover $\tilde{\mathbf{S}}$ is not uniquely determined by the topological cover $\tilde{S}$.

Definition

A Chevalley group (respectively Chevalley-Steinberg group) over a field F is a split (resp., quasisplit), absolutely almost simple, simply-connected linear algebraic group $\mathbf{G}$ over F.

Chevalley-Steinberg groups over $\mathbb{R}$

Type	Group $G = \mathbf{G}(\mathbb{R})$	$\pi_1(G)$
A_ℓ, $\ell \geq 2$	$SL_{\ell+1}(\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
B_ℓ, $\ell \geq 3$	$Spin_{\ell,\ell+1}(\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
C_ℓ, $\ell \geq 1$	$Sp_{2\ell}(\mathbb{R})$	$\mathbb{Z}$
D_ℓ, $\ell \geq 4$	$Spin_{\ell,\ell}(\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
E_6, E_7, E_8, F_4, G_2	Exceptional groups	$\mathbb{Z}/2\mathbb{Z}$
$A_{2p}^{(2)}$, $p \geq 1$	$SU_{p,p+1}(\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
$A_{2p-1}^{(2)}$, $p \geq 2$	$SU_{p,p}(\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
$D_\ell^{(2)}$, $\ell \geq 4$	$D_\ell^{(2)}(\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$
$E_6^{(2)}$	$E_6^{(2)}(\mathbb{R})$	$\mathbb{Z}/2\mathbb{Z}$

A good reference is Tits, “Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen” (LNM 40, 1967).

Covers of Chevalley-Steinberg groups over $\mathbb{R}$

Fix $\mathbf{G}$ a Chevalley-Steinberg group over $\mathbb{R}$. There exists a unique, up to unique isomorphism, nonsplit topological central extension,

$$\mu_2 \hookrightarrow \tilde{G} \twoheadrightarrow G.$$

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Theorem (Brylinski-Deligne)

There is a canonical central extension

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Theorem (Prasad-Rapinchuk, Prasad, Brylinski-Deligne)

The double cover arising from the canonical extension of Brylinski-Deligne is uniquely isomorphic to the nonsplit extension $\tilde{G}$.

Linearish covers

Nontrivial covers can yield (topologically) linear Lie groups.

Example

There exists a cover $\tilde{\mathbf{G}} = (\mathbf{G}', 2)$, where $\mathbf{G} = \mathbf{PGL_2}$, and the resulting extension

$$\mu_2 \hookrightarrow \tilde{G} \twoheadrightarrow PGL_2(\mathbb{R})$$

is isomorphic (nonuniquely) to the extension

$$\mu_2 \hookrightarrow \frac{GL_2(\mathbb{R})}{\left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} : t > 0 \right\}} \twoheadrightarrow PGL_2(\mathbb{R}).$$

N.B. $\tilde{g} \mapsto g \cdot |\det(g)|^{-1/2}$ is a faithful continuous representation.

Local fields

If F is a local field, and $\tilde{\mathbf{G}}$ is a degree n cover of a reductive group $\mathbf{G}$ over F, then one gets a topological central extension,

$$\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G,$$

where $G = \mathbf{G}(F)$. Universal extensions of Chevalley-Steinberg groups arise from such a construction.

Unramified and global properties

If F is a global field and $\tilde{\mathbf{G}}$ is a degree n cover of a reductive group $\mathbf{G}$ over F, then one gets a topological central extension,

$$\mu_n \hookrightarrow \tilde{G}_{\mathbb{A}} \twoheadrightarrow G_{\mathbb{A}} = \mathbf{G}(\mathbb{A}),$$

as well as a splitting of this extension over $\mathbf{G}(F)$.

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If $\mathcal{O}$ is the ring of integers in a nonarchimedean local field F, and $\tilde{\mathbf{G}}$ is a degree n cover of a reductive group $\mathbf{G}$ over $\mathcal{O}$, then one gets a topological central extension

$$\mu_n \hookrightarrow \tilde{G}_F \twoheadrightarrow G_F = \mathbf{G}(F),$$

as well as a splitting of this extension over $\mathbf{G}(\mathcal{O})$.

Covers

The class of covers described here has some nice properties:

- They include the most important topological central extensions, at least those that seem connected to arithmetic.
- Splitting properties allow one to study genuine unramified representations and genuine automorphic representations.
- They include some central extensions that can be studied using techniques for linear groups, but would not ordinarily get attention.
- They have a nice classification due to Brylinski and Deligne.

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- They have a nice classification due to Brylinski and Deligne.

I aim to extend the Langlands program to covers. For this purpose, I have defined an L-group associated to all covers of quasisplit groups over local and global fields.

The dual group

A quadratic form

Let $\tilde{\mathbf{G}}$ be a degree n cover of a quasisplit reductive group $\mathbf{G}$ over a field F. Let $\mathbf{T}$ be a maximal torus in a Borel subgroup $\mathbf{B} \subset \mathbf{G}$. Define

$$Y = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}), \quad X = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m).$$

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To $\tilde{\mathbf{G}}$, Brylinski and Deligne associate a Weyl- and Galois-invariant quadratic form

$$Q \colon Y \to \mathbb{Z}.$$

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Simplest case: If $\mathbf{G}$ a Chevalley-Steinberg group and $\tilde{\mathbf{G}}$ is the canonical double cover, $Q \colon Y \to \mathbb{Z}$ is the unique Weyl-invariant quadratic form such that

$$Q(\alpha^\vee) = 1 \text{ for all short coroots } \alpha^\vee.$$

Modified root data

Let $\Phi \subset X$ and $\Phi^\vee \subset Y$ the subsets of roots and coroots. Let $\Delta \subset \Phi$ and $\Delta^\vee \subset \Phi^\vee$ be the subsets of simple roots and coroots.

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Define

$$n_\alpha = \frac{n}{\mathrm{GCD}(n, Q(\alpha^\vee))} \text{ for all } \alpha \in \Phi.$$

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Define <u>modified roots and coroots</u>

$$\tilde{\alpha} = n_\alpha^{-1}\alpha, \quad \tilde{\alpha}^\vee = n_\alpha \alpha^\vee \text{ for all } \alpha \in \Phi.$$

$$\tilde{\Phi} = \{\tilde{\alpha} : \alpha \in \Phi\}, \quad \tilde{\Phi}^\vee = \{\tilde{\alpha}^\vee : \alpha \in \Phi\}.$$

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Define a modified coweight lattice

$$Y_{Q,n} = \{y \in Y : Q(y+y') - Q(y) - Q(y') \in n\mathbb{Z} \text{ for all } y' \in Y\} \subset nY.$$

Define a modified weight lattice

$$X_{Q,n} = \mathrm{Hom}(Y_{Q,n}, \mathbb{Z}) \subset n^{-1}X.$$

Modified root data

Theorem (Lusztig, Finkelberg-Lysenko, McNamara, Reich, W.)

The sextuple $(Y_{Q,n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, X_{Q,n}, \tilde{\Phi}, \tilde{\Delta})$ *is a based root datum.*

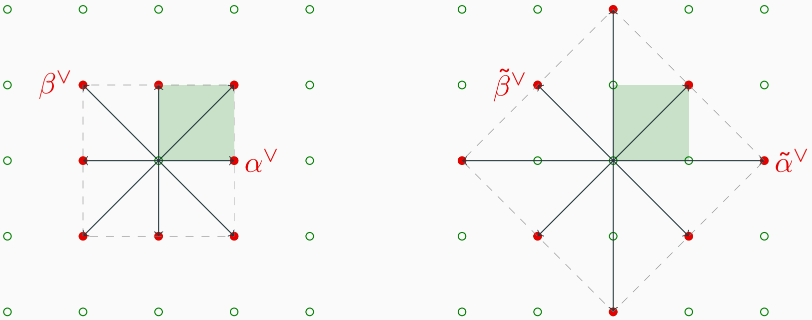

Figure 1: Modifying the root datum for the double cover of Sp_4. On the left, $Y \supset \Phi^\vee \supset \Delta^\vee$ On the right, $Y_{Q,2} \supset \tilde{\Phi}^\vee \supset \tilde{\Delta}^\vee$. In this case $Y = Y_{Q,2}$ (I call such covers "sharp").

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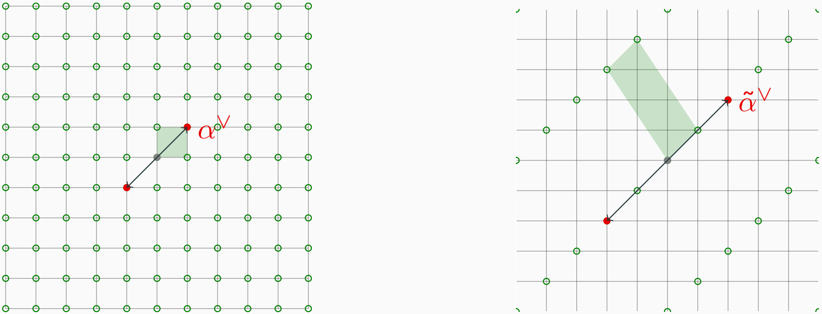

Figure 2: Modifying the root datum for a double cover of GL_2, with $\alpha^\vee = (1,1)$ and $Q(u,v) = u^2 + uv + v^2$ in standard coordinates.

The dual group

Theorem (Lusztig, Finkelberg-Lysenko, McNamara, Reich, W.)

The sextuple $(Y_{Q,n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, X_{Q,n}, \tilde{\Phi}, \tilde{\Delta})$ *is a based root datum.*

Definition

The dual group of the cover $\tilde{\mathbf{G}}$ is the pinned complex reductive group $\tilde{G}^\vee$ associated to the based root datum above.

The pinning gives a Borel subgroup and maximal torus: $\tilde{G}^\vee \supset \tilde{B}^\vee \supset \tilde{T}^\vee$. Note $\tilde{T}^\vee = \mathrm{Hom}(Y_{Q,n}, \mathbb{C}^\times)$.

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The map (a homomorphism, in fact)

$$Y_{Q,n} \to \mathbb{C}^\times, \quad y \mapsto e^{2\pi i Q(y)/n}$$

defines a 2-torsion element

$$\xi \in \tilde{Z}^\vee := Z(\tilde{G}^\vee) = \mathrm{Hom}\left(\frac{Y_{Q,n}}{Span_{\mathbb{Z}}(\tilde{\Phi}^\vee)}, \mathbb{C}^\times\right).$$

Tables

Double cover	$\tilde{G}^\vee$
$\widetilde{SL}_2(\mathbb{R})$	$^*SL_2(\mathbb{C})$
$\widetilde{SL}_3(\mathbb{R})$	$PGL_3(\mathbb{C})$
$\widetilde{SL}_4(\mathbb{R})$	$SL_4(\mathbb{C})/\mu_2$
$\widetilde{SL}_5(\mathbb{R})$	$PGL_5(\mathbb{C})$
$\widetilde{SL}_6(\mathbb{R})$	$^*SL_6(\mathbb{C})/\mu_3$
$\widetilde{Spin}_7(\mathbb{R})$	$SO_7(\mathbb{C})$
$\widetilde{Spin}_9(\mathbb{R})$	$Spin_9(\mathbb{C})$
$\widetilde{Spin}_{11}(\mathbb{R})$	$SO_{11}(\mathbb{C})$
$\widetilde{Spin}_{13}(\mathbb{R})$	$^*Spin_{13}(\mathbb{C})$

Double cover	$\tilde{G}^\vee$
$\widetilde{Sp}_{2\ell}(\mathbb{R})$	$^*Sp_{2\ell}(\mathbb{C})$
$\widetilde{Spin}_8(\mathbb{R})$	$Spin_8(\mathbb{C})$
$\widetilde{Spin}_{10}(\mathbb{R})$	$SO_{10}(\mathbb{C})$
$\widetilde{Spin}_{12}(\mathbb{R})$	$^*Spin_{12}(\mathbb{C})$
$\widetilde{E}_6(\mathbb{R})$	$E_6(\mathbb{C})/\mu_3$
$\widetilde{E}_7(\mathbb{R})$	$^*E_7(\mathbb{C})$
$\widetilde{E}_8(\mathbb{R})$	$E_8(\mathbb{C})$
$\widetilde{G}_2(\mathbb{R})$	$G_2(\mathbb{C})$
$\widetilde{F}_4(\mathbb{R})$	$F_4(\mathbb{C})$

Table 1: Table of double covers of real Chevalley groups and their dual groups. Asterisks denote the cases where the 2-torsion element ξ is nontrivial. (For quasisplit groups, Gal acts by outer automorphisms preserving ξ)

The L-group

The naïve L-group

Others (e.g., Savin, Adams-Barbasch-Paul-Trapa-Vogan, Crofts, Finkelberg-Lysenko, McNamara, Reich) related the dual $\tilde{G}^\vee$ to the parameterization of genuine representations of $\tilde{G}$.

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This suggests a naïve L-group:

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I suggest a more elaborate L-group is the natural L-group. It will be an extension,

$$\tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathsf{Gal},$$

but **without a distinguished splitting** in general.

The first twist

The first step in constructing the L-group is the "first twist".

Let σ denote complex conjugation, so $\mathsf{Gal} = \mathsf{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$.

Define a cocycle, $\mathsf{Gal} \times \mathsf{Gal} \to \tilde{Z}^\vee = Z(\tilde{G}^\vee)$ by

$$(\sigma, \sigma) \mapsto \xi.$$

$$(1, 1) \text{ and } (1, \sigma) \text{ and } (\sigma, 1) \mapsto 1.$$

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This yields a central extension,

$$\tilde{Z}^\vee \hookrightarrow E_1 \twoheadrightarrow \mathsf{Gal}\,.$$

The second Brylinski-Deligne invariant

But this isn't enough – a second twist is needed, which requires another invariant of covers due to Brylinski-Deligne.

To a cover $\tilde{\mathbf{G}}$ of a reductive group $\mathbf{G}$ over a field F, Brylinski and Deligne associate a $\mathrm{Gal}(\bar{F}/F)$-equivariant extension of groups,

$$\bar{F}^\times \hookrightarrow D \twoheadrightarrow Y,$$

where Y is the coweight lattice and $\bar{F}$ is a separable closure of F.

Chevalley-Steinberg groups

Consider $\tilde{\mathbf{G}}$ the canonical double cover of a Chevalley-Steinberg group $\mathbf{G}$ over $\mathbb{R}$. We can describe D by generators and relations.

Generators: all elements of $\mathbb{C}^\times$, and elements d_α for each simple root $\alpha \in \Delta$.

Relations: $\mathbb{C}^\times$ is contained in the center of D, and for any $\alpha, \beta \in \Delta$, $[d_\alpha, d_\beta] = (-1)^{Q(\alpha+\beta)-Q(\alpha)-Q(\beta)}$.

Inclusion of $\mathbb{C}^\times \hookrightarrow D$ and projection $d_\alpha \mapsto \alpha^\vee$ yields a central extension,

$$\mathbb{C}^\times \hookrightarrow D \twoheadrightarrow Y.$$

Galois-invariance of Q gives a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on D.

Chevalley-Steinberg groups

From a double cover $\tilde{\mathbf{G}}$, we have a central extension,

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endowed with a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action.

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Pull back to the sublattice $Y_{Q,2} \subset Y$.

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This is an *abelian* extension.

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This is an *abelian* extension.

Take $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ invariants.

$$\mathbb{R}^\times \hookrightarrow D_{Q,2}^\sigma \twoheadrightarrow Y_{Q,2}^\sigma.$$

Flipping the extension (split case)

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Definition

The <u>flipped extension</u> E_2 is the set of homomorphisms $f\colon D^\sigma_{Q,2} \to \mathbb{C}^\times$ such that

- $f(t) = 1$ for all $t \in \mathbb{R}^\times_{>0}$.
- $f(d_\alpha^{n_\alpha} \cdot r_\alpha) = 1$ for all $\alpha \in \Delta$. Here $r_\alpha = (-1)^{Q(\alpha^\vee)\cdot\binom{n_\alpha}{2}}$.

This gives an extension (not so obviously)

$$\tilde{Z}^\vee \hookrightarrow E_2 \twoheadrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

Flipping the extension (quasisplit case)

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Choose a splitting $s\colon Y_{Q,2} \to D_{Q,2}$ which satisfies

$$s(\tilde{\alpha}^\vee) = d_\alpha^{n_\alpha} \cdot r_\alpha \text{ for all } \alpha \in \Delta.$$

Then ${}^\sigma s/s \in \mathrm{Hom}(Y_{Q,2}, \mathbb{C}^\times)$, and has a square root $\sqrt{{}^\sigma s/s}$.

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There are two $\tilde{Z}^\vee$-torsors:

$$E_{2,1} = \tilde{Z}^\vee = \{f \in \mathrm{Hom}(Y_{Q,2}, \mathbb{C}^\times) : f(\tilde{\alpha}^\vee) = 1 \text{ for all } \alpha \in \Delta\}.$$

$$E_{2,\sigma} = \{f \in \mathrm{Hom}(Y_{Q,2}, \mathbb{C}^\times) : \left[f \cdot \sqrt{\frac{{}^\sigma s}{s}} \right] (\tilde{\alpha}^\vee) = 1 \text{ for all } \alpha \in \Delta\}.$$

Assembly

Define $E_2 = E_{2,1} \sqcup E_{2,\sigma}$. Then we find a short exact sequence,

$$\tilde{Z}^\vee \hookrightarrow E_2 \twoheadrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

The cocycle $(\sigma, \sigma) \mapsto \xi$ gave another short exact sequence,

$$\tilde{Z}^\vee \hookrightarrow E_1 \twoheadrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

Take the Baer sum,

$$\tilde{Z}^\vee \hookrightarrow E_1 \dot{+} E_2 \twoheadrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

Push out via the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant inclusion $\tilde{Z}^\vee \hookrightarrow \tilde{G}^\vee$ to get a short exact sequence.

$$\tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

Evidences and questions

Theorem (W. (also see Gan-Gao))

Let $\mathbf{T}$ be a split torus over a local or global field. Then there is a natural one-to-one parameterization:

$$\{ \textit{ Irreducible genuine admissible/automorphic reps of } \tilde{T} \}$$

$$\hookrightarrow \{ \textit{ Weil parameters valued in } {}^{\mathsf{L}}\tilde{T} \}$$

Evidence: Tori

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Theorem (W.)

Let $\mathbf{T}$ be a torus over $\mathbb{R}$ with $T = \mathbf{T}(\mathbb{R})$ compact. Then there is a natural one-to-one parameterization:

$$\{ \textit{Irreducible genuine characters of } \tilde{T} \}$$

$$\hookrightarrow \{ \textit{Weil parameters valued in } {}^{\mathsf{L}}\tilde{T} \}$$

Evidence: Unramified representations

Let $\tilde{\mathbf{G}}$ be a cover of a reductive group $\mathbf{G}$, defined over the ring of integers in a nonarchimedean local field.

Theorem (W. (also see Gan-Gao))

There is a natural bijective parameterization:

$$\{ \textit{ Irreducible genuine spherical reps of } \tilde{G} \} \textit{ (mod equivalence)}$$

$$\to \{ \textit{ Unramified Weil parameters valued in } {}^{\mathsf{L}}\tilde{G} \} \textit{ (mod } Ad(\tilde{G}^\vee)) \, .$$

Proof: Satake isomorphim (McNamara, WenWei Li, Gan-Gao, W.) + Parameterization for split tori + carefully tracing through Weyl-group action.

Evidence: Discrete series

Let $\tilde{\mathbf{G}}$ be a cover of a quasisplit semisimple group $\mathbf{G}$ over $\mathbb{R}$, such that $\mathbf{G}$ contains a compact maximal torus $\mathbf{T}$ over $\mathbb{R}$.

Theorem (W.)

There is a natural one-to-one parameterization:

$$\{ \textit{Discrete series reps of } \tilde{G} \} \textit{ (mod equivalence)}$$

$$\hookrightarrow \{ \textit{Discrete series Weil parameters valued in } {}^{\mathsf{L}}\tilde{G} \} \textit{ (mod } Ad(\tilde{G}^\vee)) \,.$$

Evidence: Hecke algebras

Let $\tilde{\mathbf{G}}$ be a degree n cover of a simple Chevalley group over $\mathbb{Z}_p$, type A,D,E, with $p = 3 \bmod 4$. Let $\mathbf{G}_{lin}$ be the split reductive group whose Langlands dual group is $\tilde{G}^\vee$.

Theorem (Savin, 2004)

For each Satake painting S of the Dynkin diagram, choose a square root of $(-1)^{\#S}$ in $\mathbb{C}$. This set of choices determines an isomorphism from the Iwahori Hecke algebra of $\tilde{G}$ to the Iwahori Hecke algebra of G_{lin}.

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Theorem (Gao? W.?)

For each Satake painting S of the Dynkin diagram, choose a square root of $(-1)^{\#S}$ in $\mathbb{C}$. This set of choices determines an isomorphism of L-groups from the L-group ${}^{\mathsf{L}}\tilde{G}$ of the cover to the L-group ${}^{\mathsf{L}}G_{lin} = \mathrm{Gal} \ltimes \tilde{G}^\vee$.

Question: Functoriality for linearish groups

Consider a "linearish cover" $\tilde{\mathbf{G}}$. Then genuine irreps of $\tilde{G}$ correspond to irreps of a linear group H with a specific central character.

Example

($G = PGL_2$ and $H = GL_2$)

There's a cover $\tilde{\mathbf{G}}$ for which $\mu_2 \hookrightarrow \tilde{G} \twoheadrightarrow PGL_2(\mathbb{R})$ in which

$$\tilde{G} \cong GL_2(\mathbb{R}) / \left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} : t > 0 \right\}.$$

Exercise: Pullback from $\tilde{G}$ to H should be functorial, reflected in a homomorphism of L-groups ${}^{\mathsf{L}}\tilde{G} \to {}^{\mathsf{L}}H$. This has been considered for $H = GL_2$ by Gan and Gao.

Question: General tori (over $\mathbb{R}$)

Parameterization has been accomplished for covers of split tori and covers of compact tori over $\mathbb{R}$.

Exercise: Complete the parameterization for all covers of tori over $\mathbb{R}$.

Problem: Complete the parameterization for all covers of tori over local fields.

Question: Which parameters are relevant?

The parameterizations are often one-to-one. Some Weil parameters do not correspond to irreducible genuine irreps.

Question: Which Weil parameters are "relevant"? I.e., what is the image of the parameterization map? This seems related to endoscopy for covering groups.

Question: The contragredient?

If $\tilde{\mathbf{G}}$ is a cover, there is an "opposite cover" $\tilde{\mathbf{G}}^{op}$. (For double covers, they can be taken to be the same).

If π is a genuine irreducible representation of $\tilde{G}$ (work over a local field), its contragredient is a genuine irreducible representation of $\tilde{G}^{op}$.

Question: (Adams-Vogan?) What is the corresponding map on L-groups?

Thank you...

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