Integrability of p-adic matrix coefficients

joint work with Omer Offen

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New developments in representation theory
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Setting

$\mathbf{G}$ a reductive group defined over a p-adic field F. $G = \mathbf{G}(F)$.
(π, V) an admissible representation of G.
For $v \in V$, $v^* \in V^*$,

$$m_{v,v^*}(g) = v^*(\pi(g)v), \quad g \in G$$

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Relative setting

$\mathbf{H} < \mathbf{G}$ a closed subgroup. $H = \mathbf{H}(F)$.
Symmetric case: $\mathbf{H} = \mathbf{G}^\theta$ for an F-involution θ on $\mathbf{G}$.
Relative harmonic analysis is interested in possible embeddings

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For a smooth mod center function f on G, we can try to define the integral

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Main Question - Are there local periods for π?

Given $0 \neq v^* \in (V^*)^H$, is there a *smooth* $\widetilde{v} \in \widetilde{V}$, such that

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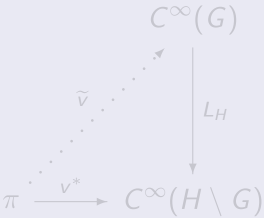
for all $v \in V$?



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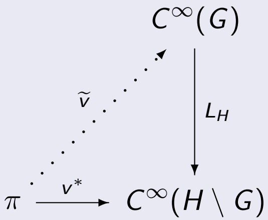

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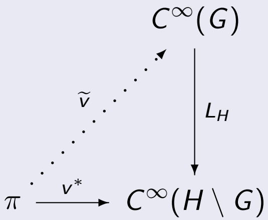

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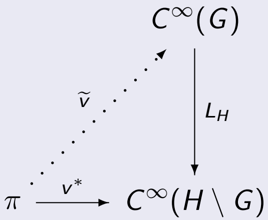

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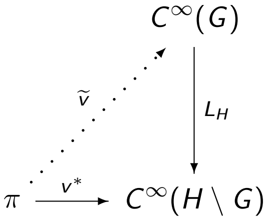

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1. Is the H-integral over $m_{v,\widetilde{v}}$ absolutely convergent?
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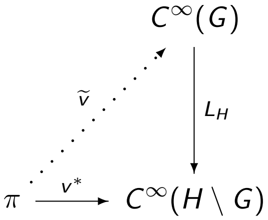

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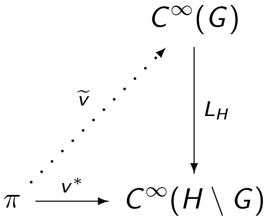

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Global motivation

A cuspidal automorphic representation $\Pi = \otimes'_v \pi_v$ of $\mathbf{G}(\mathbb{A}_k)$ (k a number field) has a canonical $\mathbf{H}(\mathbb{A}_k)$-invariant functional - the period integral: $P(\phi) = \int_{\mathbf{H}(k)\backslash\mathbf{H}(\mathbb{A}_k)} \phi(h)\, dh$.

In certain cases (not symmetric), it is expected that when $\{\pi_v\}$ are tempered, the period integral will factorize as

$$|P(\phi)|^2 = P(\phi)P(\overline{\phi}) = \prod_v{}' L_H(m_{\phi_v,\overline{\phi}_v})$$

under *suitable normalizations* of measures, where $\phi = \otimes' \phi_v \in \Pi$.

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- Lapid-Mao conjectures for the Whittaker case.
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Definitions

- A representation π is called *square-integrable* if

$$|m_{v,\widetilde{v}}| \in L^2(G/Z(G))$$

for all $v \in V$, $\widetilde{v} \in \widetilde{V}$.

- A representation π is called *tempered* if

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Strongly tempered pair

A pair (G, H) is called *strongly tempered* if any tempered irreducible representation of G is H-integrable.

Tempered distributions

The distribution L_H on $G/Z(G)$ is *tempered* if it extends as a functional to the Harish-Chandra-Schwartz space of $G/Z(G)$.

In particular, when L_H is tempered any square-integrable irreducible representation of G is H-integrable.

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Corollaries

Theorems (G.-Offen)

1. The following families of pairs are strongly tempered:

$$(GL_n, O_J),\ (U_{n,\,E/F}, O(J)),\ (Sp_{2n}, U_{n,\,E/F})$$

for any orthogonal group O_J and any unitary group $U_{n,\,E/F}$ relative to a quadratic extension E of F.

2. For the following families of pairs (G, H), the distribution L_H is tempered[a]:

$$(\mathbf{G}(E), \mathbf{G}(F)),\ (GL_n, GL_{\lfloor n/2 \rfloor} \times GL_{\lceil n/2 \rceil})\ (GL_{2n}, \mathbf{GL}_n(E))$$

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Theorems - Non-vanishing

- Sakellaridis-Venkatesh: For a strongly tempered (G, H), every (tempered) H-distinguished irreducible representation of G which is parabolically induced from a square-integrable representation has non-zero local periods.
- C. Zhang: When L_H is a tempered distribution, every H-distinguished square-integrable representation of G has non-zero local periods.

Convergence of matrix coefficients

Casselman's criterion

A representation is square-integrable (tempered), if and only if, its *exponents* are (weakly) positive.

Some structure

- Fix a maximal F-split torus $A < G$, which is θ-stable and such that $A_0 := (A^\theta)^\circ$ is a maximal F-split torus of H.
- Fix a minimal θ-stable parabolic $A < P_0 < G$ and a minimal parabolic $B < P_0$.

$$\begin{array}{llll} \Delta_G & & \subset \Sigma^G = \Sigma(A, Lie(G)) & \subset X^*(A) \\ \Delta_H & \subset \Sigma^H & \subset \Sigma^G_H = \Sigma(A_0, Lie(G)) = \Sigma^G|_{A_0} & \subset X^*(A_0) \end{array}$$

- Σ^G_H is a root system with basis $\Delta^G_H = \Delta_G|_{A_0}$. $W_H < W^G_H$
- Cartan decomposition: $H^\circ = \bigcup_{c \in C,\, a \in A_0^{+,\Delta_H}} KcaK$, where $K < G$ is a maximal compact subgroup, C is finite and

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Convergence of matrix coefficients

- Convergence of $L_H(m_{v,\tilde{v}})$ reduces to summability on A^{+,Δ_H}.
- Yet, the asymptotics of $m_{v,\tilde{v}}$ (matrix coefficient of G!) can be effectively measured only on the subcone

$$A_0 \cap A^{+,\Delta_G} = \{x \in A_0 \ : \ |\alpha(x)|_F \leq 1, \forall \alpha \in \Delta_H^G\}$$

- This can be solved by choosing coset representatives $D = [W_H^G / W_H] \subset W_H^G$ for which

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1. Problem and motivation
2. Corollaries
3. Convergence of matrix coefficients
4. Results

Exponents

- For a parabolic $B < P = MN$ with $A < M$, let $A_M < Z(M)$ be the maximal F-split torus.
- For irreducible π, $Exp(\pi, P) \subset Hom(A_M, \mathbb{C}^\times)$ is the collection of central characters appearing in subquotients of the Jacquet module $J_P(\pi)$.
- For $\chi \in Exp(\pi, P)$,

$$|\chi| \in Hom(A_M, \mathbb{R}_+^\times) \cong \mathfrak{a}_{A_M}^* := X^*(A_M) \otimes \mathbb{R}$$

- For θ-stable P we say that $\lambda \in \mathfrak{a}_{A_M}^*$ is *relatively positive* if $\lambda|_{(A_M^\theta)^\circ}$ is in the cone spanned by $\Delta_G|_{(A_M^\theta)^\circ}$.

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Main theorem

Convergence criterion (G.-Offen)

An admissible representation π of G is H-integrable, iff, for every θ-stable standard parabolic P, every $\chi \in Exp(\pi, P)$ and every $w \in D$,

$$|\chi| + \rho_w^{G/H}$$

is relatively positive. Here,

$$\rho_w^{G/H} := \delta_{P_0}^{1/2}|_{\mathfrak{a}_0^*} - w\left(\delta_{P_0^\theta}^{1/2}|_{\mathfrak{a}_0^*}\right),$$

with δ_{P_0}, $\delta_{P_0^\theta}$ being the modular characters of the groups.

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In particular, combining with Casselman's criterion, (G, H) is strongly tempered when all $\rho_w^{G/H}$ are relatively positive, and L_H is tempered when all $\rho_w^{G/H}$ are weakly relatively positive.

Thank you!