Exceptional poles of local L-factors for $GL(n)$

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We study some local L-factors $GL(n)$, via a method developed by Cogdell and Piatetski-Shapiro in [C-P]. We will look at two examples. There are other settings for which exceptional poles of L-factors are defined and useful, in particular the L-factor of pairs defined by Jacquet, Shalika and Piatetski-Shapiro in [J-P-S.83]. Cogdell and Piatetski-Shapiro introduced the notion of "exceptional pole" in this context, and provided a different proof of the multiplicativity relation of the latter L-factor. The method should also work for the Jacquet-Shalika integral representation of the exterior-square L-factor, and maybe others, and relies a lot on the theory of derivatives developped in [B-Z.76], [B-Z.77], and [Z.80].

1 General setting

- F: a p-adic field with ring of integers $\mathfrak{O}_F$.
- W'_F: the Weil-Deligne group of F.
- $\mathcal{M}_n = \mathcal{M}(n, F)$, and $G_n = GL(n, F)$. We embed $G_{n-1} \hookrightarrow G_n$ via $g \mapsto diag(g, 1)$.
- $\nu : G_n \to \mathbb{C}^*$, such that $\nu(g) = |det(g)|_F$.
- $l_n : a \in \mathcal{M}_n \mapsto l_n(a) \in F^n$ where $l_n(a)$ is the bottom row of a.
- P_n: the mirabolic group of matrices $\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix}$ for $g \in G_{n-1}$ and $x \in F^{n-1}$. If $\eta_n = (0, \dots, 0, 1)$ in F^n, then $g \in G_n$ is in P_n if and only if $\eta_n g = \eta_n$.
- N_n: the group of unipotent upper triangular matrices in G_n.
- H_n: the fixed points of an involution of G_n. More precisely we will consider:
  1. $H_n = G_n^\sigma$ for σ a Galois involution of F. We set $d = 1/2$, and $\chi_n = \mathbf{1}_{H_n}$ in this case.
  2. $H_n = G_n^\sigma$ with σ the conjugation by the matrix $diag(1, -1, \dots, (-1)^{n-1})$, it is a quasi-standard maximal Levi subgroup of type $(\lfloor (n+1)/2 \rfloor, \lfloor n/2 \rfloor)$, and we chose a certain isomorphism $h : (g_1, g_2) \mapsto h(g_1, g_2)$ from $G_{\lfloor (n+1)/2 \rfloor} \times G_{\lfloor n/2 \rfloor}$ to H_n. We set $d = 1$, and $\chi_n : h(g_1, g_2) \mapsto (\frac{\nu(g_1)}{\nu(g_2)})^{\epsilon_n/2}$ in this case, with $\epsilon_n = 0$ if n is even, and 1 if n is odd.
- ψ a non trivial character of $(F, +)$ trivial on $F_0 = F^\sigma$ in Case 1.
- θ_n: the non degenerate character $g \mapsto \psi(g_{1,2} + \dots + g_{n-1,n})$ of N_n, we have $\theta_n | N_n \cap H_n = \mathbf{1}$.
- q: the residual cardinality of F in Case 1, and of F_0 in Case 2.

2 Local L-factors

Let $\pi = LQ(\tau) = LQ(\delta_1, \dots, \delta_r)$ be an irreducible representation of G_n, which is the Langlands quotient of a representation $\tau = \delta_1 \times \cdots \times \delta_r$ induced of Langlands type of G_n (a "standard module"). Set

$$W(\pi, \theta_n) = W(\tau, \theta_n).$$

Set $V_n = l_n(\mathcal{M}_n^\sigma)$. For $W \in W(\pi, \theta_n)$, and $\Phi \in \mathcal{C}_c^\infty(V_n)$, we set

$$I_n(s, W, \Phi) = \int_{N_n \cap H_n \backslash H_n} W(h)\Phi(l_n(h))\chi_n(h)\nu(h)^{ds} dh.$$

There is $r_\pi \in \mathbb{R}$ such that all such integrals converge absolutely for $\mathrm{Re}(s) \geq r_\pi$. They generate a fractional ideal of $\mathbb{C}[q^{\pm s}]$, it has a unique generator which is an Euler factor $(1/P_{H,\pi}(q^{-s})$ with P_π in $\mathbb{C}[X]$ such that $P(0) = 1)$. We denote by $L_H(s, \pi)$ this Euler factor. In Case 1, it is the Asai L-factor defined by Flicker ([F]), and in Case 2 the Bump-Friedberg L-factor ([B-F]). The integrals I_n satisfy a functional equation relating s and $1-s$ in Case 1, and s and $1/2-s$ in Case 2.

We set

$$I_{n,(0)}(s, W) = \int_{N_{n-1} \cap H_{n-1} \backslash H_{n-1}} W(h)\chi_{n-1}(h)\nu(h)^{d(s-1)} dh.$$

There is $t_\pi \in \mathbb{R}$ such that all such integrals converge for $\mathrm{Re}(s) \geq t_\pi$. They generate a fractional ideal of $\mathbb{C}[q^{\pm s}]$, it has a unique generator which is an Euler factor. We denote by $L_{H,(0)}(s, \pi)$ this Euler factor. As the vector space spanned by the integrals $I_{n,(0)}(s, W)$ is the same as that spanned by the integrals $I_n(s, W, \Phi)$ with $\Phi(0) = 0$, the factor $L_{H,(0)}(s, \pi)$ divides $L_H(s, \pi)$.

When $\pi = \rho$ is cuspidal, it follows from the fact that the Whittaker functions in $W(\rho, \theta_n)$ restrict to P_n with compact support mod N_n, that $L_{H,(0)}(s, \rho) = 1$.

We call $\pi = LQ(\tau)$ generic if $\pi = \tau$. Suppose that θ_n is of level zero, and that ϕ is the Langlands parameter of a generic unramified representation π. Then with a good normalisation of the right invariant measure on $N_n \backslash H_n$, if W_0 is the normalised spherical Whittaker function in $W(\pi, \theta_n)$, and Φ_0 the characteristic function of $l_n(\mathcal{M}(n, \mathfrak{O}_F)^\sigma)$, it is known that $I_n(s, W_0, \Phi_0)$ is equal to the Asai L-function

$$L_H(s, \phi) := L(s, M_{W'_F}^{W'_{F_0}}(\phi))$$

in Case 1 (with $M_{W'_F}^{W'_{F_0}}$ the Asai transfer), and to

$$L_H(s, \phi) := L(s + 1/2, \phi)L(2s, \Lambda^2(\phi))$$

in Case 2. Those Galois factors satisfy natural multiplicativity relations which reflect the way that $M_{W'_F}^{W'_{F_0}}$ and Λ^2 behave with respect to direct sum. In fact, it is known that

$$L_H(s, \phi) = L_H(s, \pi)$$

for any irreducible representation ϕ of G_n with Langlands parameter ϕ, and one needs to establish the multiplicativity relation of L_H to prove this.

3 Exceptional poles and distinction

Let $\pi = LQ(\tau)$ with $\tau = \delta_1 \times \cdots \times \delta_r$. We set

$$L_H^{(0)}(s, \pi) = \frac{L_H(s, \pi)}{L_{H,(0)}(s, \pi)}.$$

The factor $L_{H,(0)}$ has simple poles, which are called the *exceptional poles* of $L_H(s,\pi)$.

In particular, when $\pi = \rho$ is cuspidal, one has $L_H(s,\rho) = L_H^{(0)}(s,\rho)$, and all the poles of $L_H(s,\rho)$ are exceptional.

Proposition 3.1. *If s_0 is an exceptional pole of $L_H(s,\pi)$, then τ is $(H_n, \chi_n^{-1}\nu^{-ds_0})$-distinguished (we will often say $\chi_n^{-1}\nu^{-ds_0}$-distinguished, or simply distinguished if $\chi_n^{-1}\nu^{-ds_0} = \mathbf{1}_{H_n}$), i.e. $Hom_{H_n}(\tau, \chi_n^{-1}\nu^{-ds_0}) \neq \{0\}$.*

Proof. Of the second statement. We already said that the vector-space generated by the integrals $I_{n,(0)}(s,W)$ is the same as that generated by the integrals $I_{n,(0)}(s,W,\Phi)$ with $\phi \in \mathcal{C}_{c,0}^{\infty}(V_n)$ (i.e. $\Phi(0)=0$). Let s_0 be a pole of $L_H(s,\pi)$ of order e. Write the Laurent expansion:

$$I(s,W,\phi) = \frac{B_{s_0}(W,\Phi)}{(1-q^{(s_0-s)})^e} + \dots .$$

By definition, the bilinear form is nonzero, and for all $h \in H_n$, it satisfies

$$B_{s_0}(\rho(h)W, \rho(h)\Phi) = \chi_n^{-1}\nu^{-ds_0}(h) B_{s_0}(W,\Phi).$$

To say that s_0 is a pole of $L_H^{(0)}(s,\pi)$ is equivalent to say that B_{s_0} vanishes on $W(\pi,\theta) \times \mathcal{C}_{c,0}^{\infty}(V_n)$. If it is the case then

$$B_{s_0} = \lambda_{s_0} \otimes (\Phi \mapsto \Phi(0)),$$

with $\lambda_{s_0} \in Hom_{H_n}(\pi, \chi_n^{-1}\nu^{-ds_0}) - \{0\}$. □

For $u \in (\frac{\mathbb{C}}{2i\pi\mathbb{Z}/Ln(q_F)})^r$, we write

$$\tau_u = \nu^{u_1}\delta_1 \times \dots \times \nu^{u_r}\delta_r.$$

In Case 1, a converse of the last statement above is true when π is generic. It should also be true in Case 2. More precisely, one has the following characterisation of distinction.

Theorem 3.1. *Let π be a generic representation of G_n which is $\chi_n^{-1}\nu^{-ds_0}$-distinguished. If*

$$Hom_{H_n}(\pi, \chi_n^{-1}\nu^{-ds_0}) = Hom_{H_n \cap P_n}(\pi, \chi_n^{-1}\nu^{-ds_0}), \tag{1}$$

then $L_H(s,\pi)$ has an exceptional pole at s_0. By a theorem of Ok ([Ok.97]), in Case 1., Equality (1) is always satisfied. In Case 2., it is proved that Equality (1) is satisfied for discrete series, and when $r \geq 2$, for τ_u at least when u is in general position with respect to π (i.e. belongs to a well-chosen non empty Zariski open subset of $(\frac{\mathbb{C}}{2i\pi\mathbb{Z}/Ln(q_F)})^r$ depending on π but not on s_0).

In all cases, at least when u is in general position, one has

$$L_H^{(0)}(s,\tau_u) = \prod_{s_0,\ \mathrm{Hom}(\tau_u, \chi_n^{-1}\nu^{-ds_0}) \neq 0} \frac{1}{1-q^{(s_0-s)}}.$$

It follows that one can get an explicit expression of $L_H^{(0)}(s,\tau_u)$ in terms of the inducing data of τ_u as soon as one has a classification of generic distinguished representations in terms of this inducing data.

Theorem 3.2 ([M.11],[M.15])**.** *Let π be a generic representation of G_n, it is distinguished if and only if π is obtained as an induced representation*

$$(\delta_1^{\vee} \times \delta_1^{\sigma}) \times \dots \times (\delta_s^{\vee} \times \delta_s^{\sigma}) \times \delta_{s+1} \times \dots \times \delta_t,$$

with δ_i distinguished for $i > s$.

Remark 3.1. In Case 2., when n is even, then it is possible to use the theorem above to show that H_n-distinguished generic representations are the generic representations admitting a Shalika model (see [M.15-2]). Kaplan has proved an analogous theorem for generic representations of G_n distinguished by the tensor product of a pair of exceptional representations of a metaplectic cover of G_n ([K.15]).

Example 3.1. We restrict to Case 1, in particular $\chi_n = \mathbf{1}_{H_n}$ and $d = 1/2$. Let ρ_1 and ρ_2 be two unlinked cuspidal representations, in this case $u = 0$ is in general position with respect to

$$\pi = LQ(\rho_1, \rho_2) = \rho_1 \times \rho_2.$$

Then

$$L_H^{(0)}(s, \rho_1 \times \rho_2) = lcm(gcd(L_H(s, \rho_1), L_H(s, \rho_2)), L(2ds, \rho_1, \rho_2^\sigma)),$$

where $L(s, \rho_1, \rho_2^\sigma)$ is the Rankin-Selberg L-factor of the pair (ρ_1, ρ_2^σ). Indeed, both Euler factors have simple poles, so it is enough to check that they are the same. But s_0 is a pole of $L_H^{(0)}(s, \rho_1 \times \rho_2)$ if and only if $\rho_1 \times \rho_2$ is $\nu^{-s_0/2}$-distinguished, which is the same as:

$$\nu^{s_0/2}(\rho_1 \times \rho_2) = \nu^{s_0/2}\rho_1 \times \nu^{s_0/2}\rho_2$$

is distinguished. But according to the theorem above, it means that either both $\nu^{s_0/2}\rho_1$ and $\nu^{s_0/2}\rho_2$ are distinguished, or that $\nu^{-s_0/2}\rho_1^\vee \simeq \nu^{s_0/2}\rho_2^\sigma$. The first condition is equivalent to $L_H(s, \rho_1)$ and $L_H(s, \rho_2)$, i.e. their gcd having a pole at s_0, and the second to $L_H^{(0)}(s, \rho_1, \rho_2^\sigma)$ having a pole at s_0.

4 Computation of $L_{H,(0)}(s, \pi)$ and the multiplicativity relation

The integrals which define $L_{H,(0)}(s, \pi)$ only depend on the restrictions of the Whittaker functions $W \in W(\pi, \theta_n) = W(\tau, \theta_n)$ to G_{n-1}, or equivalently to P_n. Now by the theory of derivatives (Gelfand-Kazhdan, Bernstein-Zelevinsky), it is known that $\tau_{|P_n}$ admits a filtration with each subquotient induced to P_n by a representation $\tau^{(k)}$ of G_{n-k} of finite length. Following Cogdell and Piatetski-Shapiro, one shows that when u is in general position (with respect to π), τ_u is generic, all the derivatives of τ_u are semi-simple and one has the formula:

$$L_{(0),H}(s, \tau_u) = lcm(L_H(s, \tau_{u,i_k})) \tag{2}$$

for $n - 1 \geq k \geq 1$, and τ_{u,i_k} varying amongst the simple factors of the nonzero derivatives $\tau_u^{(k)}$, which are explicitely known thanks to the work of Bernstein and Zelevinsky.

Hence $L_{(0),H}(s, \tau_u)$ can be computed inductively, and reassembling the expression obtained for $L_{(0),H}(s, \tau_u)$, and $L_H^{(0)}(s, \tau_u)$, one gets:

Theorem 4.1. *Let π be an irreducible representation of G_n, for u in general position with respect to π, then one has:*

$$L_H(s, \tau_u) = \prod_k L_H(s, \nu^{u_k}\delta_k) \prod_{i<j} L_H(2ds, \nu^{u_i}\delta_i, \nu^{u_j}\delta_j^\sigma).$$

We continue our example.

Example 4.1. $\pi = \rho_1 \times \rho_2$ with the ρ_i's cuspidal, unlinked, and not isomorphic to one another. In this case $u = 0$ belongs to the set of points in general position with respect to π where the formula for $L_{(0),H}(s, \pi)$ applies, and the simple factors of the non zero derivatives $\pi^{(k)}$ for $n - 1 \geq k \geq 1$ are ρ_1 and ρ_2. Hence

$$L_{(0),H}(s, \pi) = lcm(L_H(s, \rho_1), L_H(s, \rho_2)).$$

But we already saw that

$$L_H^{(0)}(s,\pi) = lcm(gcd(L_H(s,\rho_1), L_H(s,\rho_2)), L(s,\rho_1,\rho_2^\sigma)).$$

The assumption that ρ_1 is not ismorphic to ρ_2 implies that $gcd(L_H(s,\rho_1), L_H(s,\rho_2))$ and $L(s,\rho_1,\rho_2^\sigma)$ are coprime, hence

$$L_H^{(0)}(s,\pi) = gcd(L_H(s,\rho_1), L_H(s,\rho_2))L(s,\rho_1,\rho_2^\sigma),$$

so that

$$\begin{aligned} L_H(s,\pi) &= lcm(L_H(s,\rho_1), L_H(s,\rho_2))gcd(L_H(s,\rho_1), L_H(s,\rho_2))L(s,\rho_1,\rho_2^\sigma) \\ &= L_H(s,\rho_1)L_H(s,\rho_2))L(s,\rho_1,\rho_2^\sigma). \end{aligned}$$

To obtain the multiplicativity relation for π, following again Cogdell and Piatetski-Shapiro, one uses a theorem of Bernstein to show that the integrals $I_n(s, W_u, \phi)$ for W_u in $W(\tau_u, \theta_n)$ corresponding to a flat section, are rational in the variables q^{-s} and q^{-u_i}. Then, following Jacquet, Shalika, and Piatetski-Shapiro, using the local functional equation, it is possible to show that the multiplicativity relation holds for all u such that τ_u is induced of Langlands type, in particular it holds for $L_H(s,\pi)$.

5 Multiplicativity relation for discrete series

We treat Case 1. We denote by $\eta_{E/F}$ the quadratic character of F^* attached to the extension E/F. We recall the following result. (Here we cheat a little, as the proof of the result below uses the multiplicativity relation of L_H, which is what we want to prove using this result. However, it should be possible to prove the result in question directly, we won't try to justify this claim.)

Theorem 5.1. *Let ρ be a cuspidal representation of G_r, and $St(k,\rho)$ the irreducible quotient of*

$$\nu^{\frac{1-k}{2}}\rho \times \cdots \times \nu^{\frac{k-1}{2}}\rho.$$

The representation $St(k,\rho)$ is distinguished if and only if ρ is $\eta_{E/F}^{k-1}$-distinguished.

Call η an extension of $\eta_{E/F}$ to E^*. The result above implies the following multiplicativity relation:

Theorem 5.2.

$$L_H(s, St(k,\rho)) = \prod_{l=0}^{k-1} L_H(s+l, \eta^{k-1-l}\rho).$$

Proof. The theorem 5.1 implies that he generalised Steinberg representation $St(k,\rho)$ is $\nu^{-s_0/2}$-distinguished if and only if $\eta^{k-1}\rho$ is $\nu^{-s_0/2}$-distinguished. Hence the factor $L_H^{(0)}(s, St(k,\rho))$ has a pole at s_0 if and only if $L_H^{(0)}(s,\eta^{k-1}\rho) = L_H(s,\eta^{k-1}\rho)$ has a pole at s_0. As both factors have simple poles, they are equal, and we deduce the equality

$$L_H^{(0)}(s, St(k,\rho)) = L_H(s,\eta^{k-1}\rho).$$

On the other hand, $u = 0$ is in general position with respect to $St(k,\rho)$, and the derivatives $St(k,\rho)^{(d)}$ of $St(k,\rho)$ for $n-1 \geq d \geq 1$ are either zero, or irreducible, and the nonzero ones are the representations $\nu^{l/2}St(k-l,\rho)$ for $l = 1,\ldots,k-1$. By induction hypothesis, one thus has

$$L_H(s, \nu^{l/2}St(k-l,\rho)) = \prod_{i=0}^{k-l-1} L_H(s+i+l, \eta^{k-l-1-i}\rho) = \prod_{i=l}^{k-1} L_H(s+i, \eta^{k-1-i}\rho),$$

and thanks to Equality (2), we know that $L_{H,(0)}(s,\pi)$ is the lcm of the factors above. This gives

$$L_{H,(0)}(s, St(k-l,\rho)) = \prod_{l=1}^{k-1} L_H(s+l, \eta^{k-1-l}\rho).$$

Finally, one gets:

$$L_H(s, St(k,\rho)) = L_H^{(0)}(s, St(k,\rho)) L_{H,(0)}(s, St(k,\rho)) = \prod_{l=0}^{k-1} L_H(s+l, \eta^{k-1-l}\rho).$$

□

Remark 5.1. The multiplicativity relation of the integral representation of the Asai γ-factor, exterior and symmetric square, as well as its stability under highly ramified twists are not proved.

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