Local gamma factor, Converse Theorem and Related problems

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Notations

$\mathbb{F}_q$: a finite field of q elements.

F: a p-adic field.

K : either be a finite field $\mathbb{F}_q$ or a p-adic field F.

B_n: the standard Borel subgroup of GL_n.

U_n: the unipotent radical of B_n.

Z_n: the center of GL_n.

Notations

Let ψ be a fixed nontrivial additive character of K.

Definition

A character ψ' *of* U_n *is called* **non-degenerate** *if*

$$\psi'(u) = \psi(\sum_{i=1}^{n-1} a_i u_{i,i+1}), \text{ for } u = (u_{i,j}) \in \mathrm{U}_n,$$

for some $a_i \in K^\times$.

ψ_n*: the* **standard non-degenerate** *character given by*

$$\psi_n(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1}), \text{ for } u = (u_{i,j}) \in \mathrm{U}_n.$$

Definition

*Let π be an irreducible representation of GL_n. Given a non-degenerate character ψ' of U_n, we call π ψ'-**generic** if*

$$\dim \mathrm{Hom}_{\mathrm{GL}_n}(\pi, \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{GL}_n}\psi') \neq 0.$$

Definition

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$$\dim \mathrm{Hom}_{\mathrm{GL}_n}(\pi, \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{GL}_n}\psi') \neq 0.$$

If π is ψ'-generic, then it is also ψ''-generic for any other non-degenerate character ψ'' of U_n.

Uniqueness of Whittaker model

Theorem

Let π *be an irreducible representation of* GL_n. *Then*

$$\dim \mathrm{Hom}_{\mathrm{U}_n}(\pi|_{\mathrm{U}_n}, \psi_n) = \dim \mathrm{Hom}_{\mathrm{GL}_n}(\pi, \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{GL}_n} \psi_n) \leq 1.$$

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When π is generic, the above Hom-space is of dimension one.

Space of Whittaker model

Let $\ell_{\psi_n} \in \mathrm{Hom}_{\mathrm{U}_n}(\pi|_{\mathrm{U}_n}, \psi_n)$ be a nonzero Whittaker functional of π.

Define $W_v(g) := \ell_{\psi_n}(\pi(g)v)$, for $v \in V_\pi$.

$W_v \in \mathrm{Ind}_{\mathrm{U}_n}^{\mathrm{GL}_n} \psi_n$ is called the Whittaker function attached to the vector v.

The subspace generated by all Whittaker functions $W_v(g)$ is unique and will be denoted by $\mathcal{W}(\pi, \psi_n)$. This space is called the Whittaker model of π.

Cuspidal representations are generic

For $W_v \in \mathcal{W}(\pi, \psi)$, denote by $\widetilde{W_v}$ the function on GL_n given by

$$\widetilde{W_v}(g) = W_v(w_n({}^t g^{-1})),\ g \in \mathrm{GL}_n,$$

where w_n is the longest Weyl element of GL_n,

with 1's on the second diagonal and zeros elsewhere.

Then

$$\widetilde{W_v} \in \mathcal{W}(\tilde{\pi}, \psi_n^{-1}),$$

where $\tilde{\pi}$ denotes the representation contragredient to π.

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Then

$$\widetilde{W_v} \in \mathcal{W}(\tilde{\pi}, \psi_n^{-1}),$$

where $\tilde{\pi}$ denotes the representation contragredient to π.

Theorem

Any irreducible cuspidal representation of GL_n is generic.

Let $n,\ t \geq 1$ be integers.

Let π be an irreducible generic representations of $\mathrm{GL}_n(F)$, with central characters ω_π.

Let τ be irreducible generic representations of $\mathrm{GL}_t(F)$, with central characters ω_τ.

Let $W_\pi \in \mathcal{W}(\pi, \Phi_n)$ be a Whittaker function of π.

Let $W_\tau \in \mathcal{W}(\tau, \Phi_t^{-1})$ be a Whittaker function of τ.

We assume that $n > t$.

For $g \in \mathrm{GL}_n(F)$, we denote by R_g the right translation action of g on functions from $\mathrm{GL}_n(F)$ to $\mathbb{C}$.

Let $w_{n,t} = \begin{pmatrix} \mathrm{I}_t & 0 \\ 0 & w_{n-t} \end{pmatrix}$.

Zeta Integrals

For $n-t-1 \geq j \geq 0$, a local zeta integral for the pair (π, τ) is defined by

$$\mathcal{Z}(W_\pi, W_\tau, s; j) := \int_g \int_x W_\pi \begin{pmatrix} g & 0 & 0 \\ x & \mathrm{I}_{n-t-1-j} & 0 \\ 0 & 0 & \mathrm{I}_{j+1} \end{pmatrix} W_\tau(g) |\det g|^{s-\frac{n-t}{2}} dx dg,$$

where the integration in the variable g is over $\mathrm{U}_t(F)\backslash\mathrm{GL}_t(F)$ and the integration in the variable x is over $\mathrm{Mat}_{(n-t-j-1)\times t}(F)$.

Jacquet, Piatetski-Shapiro, and Shalika proved in [JP-SS83] the following theorem.

Theorem ([JP-SS83, Section 2.7])

1. *Each integral* $\mathcal{Z}(W_\pi, W_\tau, \Phi, s; j)$ *is absolutely convergent for* $\mathrm{Re}(s)$ *sufficiently large and is a rational function of* q^{-s}*. More precisely, for fixed* j*, the integrals* $\mathcal{Z}(W_\pi, W_\tau, s; j)$ *span a fractional ideal (independent of* j*)*
$$\mathbb{C}[q^s, q^{-s}]L(s, \pi \times \tau)$$
of the ring $\mathbb{C}[q^s, q^{-s}]$*, where the local* L*-factor* $L(s, \pi \times \tau)$ *has the form* $P(q^s)^{-1}$*, with* $P \in \mathbb{C}[x]$ *and* $P(0) = 1$*.*
2. *For* $n - t - 1 \geq j \geq 0$*, there is a factor* $\epsilon(s, \pi \times \tau, \psi)$ *independent of* j*, such that*
$$\frac{\mathcal{Z}(R_{w_{n,r}}\widetilde{W}_\pi, \widetilde{W}_\tau, 1-s; n-t-j-1)}{L(1-s, \tilde{\pi} \times \tilde{\tau})} = \omega_\tau(-1)^{n-1}\epsilon(s, \pi \times \tau, \psi)\frac{\mathcal{Z}(W_\pi, W_\tau, s; j)}{L(s, \pi \times \tau)}.$$

Gamma factors for $\mathrm{GL}_n(F)$

The local gamma factor attached to a pair of representations π and τ is defined by

$$\Gamma(s, \pi \times \tau, \psi) = \epsilon(s, \pi \times \tau, \psi) \frac{L(1-s, \tilde{\pi} \times \tilde{\tau})}{L(s, \pi \times \tau)}. \tag{1.1}$$

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The functional equation in Part (ii) of Theorem 1.5 can be rewritten

$$\mathcal{Z}(R_{w_{n,t}}\widetilde{W}_\pi, \widetilde{W}_\tau, 1-s; n-t-j-1) \tag{1.2}$$

$$= \omega_\tau(-1)^{n-1}\Gamma(s, \pi \times \tau, \psi_F)\mathcal{Z}(W_\pi, W_\tau, s; j).$$

Gamma factors for $\mathrm{GL}_n(\mathbb{F}_q)$

Roditty in her thesis considered the finite field analogue of zeta integral and gamma factors.

Theorem

Let π *be an irreducible cuspidal representation of* $\mathrm{GL}_n(\mathbb{F}_q)$ *and* τ *an irreducible generic representation of* $\mathrm{GL}_t(\mathbb{F}_q)$*, with* $n > t$*. Then there exists a complex number* $\gamma(\pi \times \tau, \psi)$ *such that*

$$\gamma(\pi \times \tau, \psi) q^{tj} \sum_{m \in \mathrm{U}_t \backslash \mathrm{GL}_t(\mathbb{F}_q)} \sum_{x \in \mathrm{M}_{n-t-j-1,t}} W_\pi \begin{pmatrix} m & 0 & 0 \\ x & \mathrm{I}_{n-t-j-1} & 0 \\ 0 & 0 & \mathrm{I}_{j+1} \end{pmatrix} W_\tau(m)$$

$$= \sum_{m \in \mathrm{U}_j \backslash \mathrm{GL}_t(\mathbb{F}_q)} \sum_{y \in \mathrm{M}_{t,j}} W_\pi(\begin{pmatrix} 0 & \mathrm{I}_{n-t-j} & 0 \\ 0 & 0 & \mathrm{I}_j \\ m & 0 & y \end{pmatrix}) W_\tau(m),$$

for all $0 \leq j \leq n-t-1$, $W_\pi \in \mathcal{W}(\pi, \psi_n)$ *and* $W_\tau \in \mathcal{W}(\tau, \psi_t^{-1})$.

$n \times m$ Local Converse Problem

Question ($n \times m$ Local Converse Problem for $\mathrm{GL}_n(F)$)

Let π_1, π_2 be two irreducible generic representations of $\mathrm{GL}_n(F)$ with the same central character. If the (local) γ-factors $\gamma(\pi_1 \times \tau, \psi)$ and $\gamma(\pi_2 \times \tau, \psi)$ agree for any irreducible generic representation τ of $\mathrm{GL}_t(F)$, with $t = 1,\ 2,\ \cdots,\ m$, can we deduce that π_1 and π_2 are isomorphic?

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Question ($n \times m$ Local Converse Problem for $\mathrm{GL}_n(\mathbb{F}_q)$)

Let π_1, π_2 be two irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$ with the same central character. If the (local) γ-factors $\gamma(\pi_1 \times \tau, \psi)$ and $\gamma(\pi_2 \times \tau, \psi)$ agree for any irreducible generic representation τ of $\mathrm{GL}_t(\mathbb{F}_q)$, with $t = 1, 2, \cdots, m$, can we deduce that π_1 and π_2 are isomorphic?

Jacquest's conjecture on LCP

Conjecture (Jacquest's conjecture for p-adic case)

Let π_1 and π_2 be irreducible smooth **generic** *representations of $\mathrm{GL}_n(F)$ with the same central character. Assume that the local gamma factors $\Gamma(s, \pi_1 \times \tau, \psi)$ and $\Gamma(s, \pi_2 \times \tau, \psi)$ are equal as functions in the complex variable $s \in \mathbb{C}$, for all irreducible representations τ of $\mathrm{GL}_t(F)$, $1 \leq t \leq [\frac{n}{2}]$. Then π_1 and π_2 are isomorphic.*

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Conjecture (Finite field analogue of Jacquest's conjecture)

Let π_1 and π_2 be irreducible smooth **cuspidal** *representations of* $\mathrm{GL}_n(\mathbb{F}_q)$ *with the same central character. Assume that the local gamma factors* $\gamma(\pi_1 \times \tau, \psi)$ *and* $\gamma(\pi_2 \times \tau, \psi)$ *are equal for all irreducible representations* τ *of* $\mathrm{GL}_t(\mathbb{F}_q)$, $1 \leq t \leq [\frac{n}{2}]$. *Then* π_1 *and* π_2 *are isomorphic.*

What is known?

Henniart's proved $n \times (n-1)$ Local Converse Theorem for $\mathrm{GL}_n(F)$ in 1993.

Jeff, Jiang-Ping Chen proved in $n \times (n-2)$ Local Converse Theorem for $\mathrm{GL}_n(F)$ in 1996.

Edva-Aida Roditty proved the $n \times (n-1)$ and $n \times (n-2)$ Local Converse Theorem for $\mathrm{GL}_n(\mathbb{F}_q)$.

Nien proved the $n \times [\frac{n}{2}]$ Local Converse Theorem for $\mathrm{GL}_n(\mathbb{F}_q)$ in 2014.

In the joint work with Dihua Jiang and Shaun Stevens, we gave a formulation and used this formulation to verify Jacquet's conjecture on Local Converse Theorem for many cases, in 2015.

$\mathfrak{o}$: the ring of integers in F.

$\mathfrak{p}$: the prime ideal in $\mathfrak{o}_F$.

$\mathfrak{k}$: the residue field of F, of cardinality q.

$\mathrm{c}-\mathrm{Ind}$: the compact induction functor.

Z_n : the center of $\mathrm{GL}_n(F)$.

$\mathrm{K}_n := \mathrm{GL}_n(\mathfrak{o})$: the maximal compact subgroup of $\mathrm{GL}_n(F)$

$\mathfrak{P}_n := \mathrm{I}_n + \mathrm{Mat}_n(\mathfrak{p})$.

Stability of Gamma Factors

Proposition ([JS85])

Let π be an irreducible generic representation of $\mathrm{GL}_n(F)$ with $n \geq 2$. Then there exits m_π such that, for any character χ of $F^\times$ of conductor $m \geq m_\pi$ and any $c \in \mathfrak{p}^{-m}$ satisfying $\chi(1+x) = \psi(cx)$, for $x \in \mathfrak{p}^{[\frac{m}{2}]+1}$, we have

$$L(s, \pi \times \chi) = 1 \text{ and } \epsilon(s, \pi \times \chi, \psi) = \omega_\pi(c)^{-1} \epsilon(s, 1 \times \chi, \psi)^n.$$

$n \times 1$ Gamma Factors determine the central character

Corollary ([JNS15])

Let π_1, π_2 be irreducible generic representations of $\mathrm{GL}_n(F)$. If their local gamma factors $\Gamma(s, \pi_1 \times \chi, \psi_F)$ and $\Gamma(s, \pi_2 \times \chi, \psi_F)$ are equal as functions in the complex variable s, for any character χ of $F^\times$, then they possess the same central character.

Proof.

For $i = 1, 2$, let $m_{\pi_i}, m_{\tilde{\pi}_i}$ be the numbers given by Proposition 2.3 and put $m_0 = \max\{m_{\pi_i}, m_{\tilde{\pi}_i} \mid i = 1, 2\}$. For χ a character of $F^\times$ of conductor $m \geq m_0$, we have $\epsilon(s, \pi_i \times \chi, \psi) = \Gamma(s, \pi_i \times \chi, \psi)$, by stability of Gamma factors..

For any $c \in \mathfrak{p}^{-m} \setminus \mathfrak{p}^{1-m}$, with $m \geq m_0$, there exists a character χ_c character of conductor m such that $\chi_c(1 + x) = \psi(cx)$, for $x \in \mathfrak{p}_F^{[\frac{m}{2}]+1}$; thus Proposition 2.3 implies

$$\omega_{\pi_1}(c) = \omega_{\pi_2}(c).$$

Since any element of $F^\times$ can be expressed as the quotient of two elements of valuation at most $-m$, we deduce that $\omega_{\pi_1} = \omega_{\pi_2}$. $\square$

$n \times [\frac{n}{2}]$ Local Converse Theorem for p-adic GL_n

Adrian, M.; Liu, B.; Stevens, S. and Xu, P. verified Jacquet's conjecture for $\mathrm{GL}_p(F)$ for prime p in 2015.

$n \times [\frac{n}{2}]$ Local Converse Theorem for p-adic GL_n

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Recently, Liu, B. and Jacquet, H. using analytic approaches and J. Chai using Bessel distributions independently settled this long standing conjecture.

Theorem

Let π_1 and π_2 be irreducible smooth **generic** *representations of $\mathrm{GL}_n(F)$ with the same central character. Assume that the local gamma factors $\Gamma(s, \pi_1 \times \tau, \psi)$ and $\Gamma(s, \pi_2 \times \tau, \psi)$ are equal as functions in the complex variable $s \in \mathbb{C}$, for all irreducible representations τ of $\mathrm{GL}_t(F)$, $1 \leq t \leq [\frac{n}{2}]$. Then π_1 and π_2 are isomorphic.*

Sharpness of the bound $[\frac{n}{2}]$ for cuspidal representations of $\mathrm{GL}_n(F)$

In 2015, Adrian, M.; Liu, B.; Stevens, S. and Tam, K.-F. showed $[\frac{n}{2}]$ is the sharp bound for necessary twisting in Local Converse Theorem for a pair of supercuspidal representations of p-adic GL_n if n is a prime.

However, for special family of supercuspidal representations, the upper bound may be lower.

C. Bushnell and G. Henniart found: for simple supercuspidal representations(i.e. supercuspidal representations with minimal positive depth), the upper bound may be lowered to 1.

Bessel functions for $\mathrm{GL}_n(\mathbb{F}_q)$

Proposition

Let π *be an irreducible generic representation of* $\mathrm{GL}_n(\mathbb{F}_q)$ *and* χ_π *its character. Define*

$$\mathcal{B}(g) = |\mathrm{U}_n(\mathbb{F}_q)|^{-1} \sum_{u \in \mathrm{U}_n} \psi_n(u^{-1})\chi_\pi(gu), \text{ for } g \in \mathrm{GL}_n(\mathbb{F}_q).$$

Then

1. $\mathcal{B} \in \mathcal{W}(\pi, \psi_n)$.
2. $\mathcal{B}(\mathrm{I}_n) = 1$.
3. $$\mathcal{B}(u_1 g u_2) = \psi_n(u_1)\psi_n(u_2), \text{ for all } u_1,\ u_2, \in \mathrm{U}_n, g \in \mathrm{GL}_n(\mathbb{F}_q).$$

The unique function given in the above theorme is called the Bessel functions for π.

Proposition ([Ro])

Let π *be an irreducible cuspidal representation of* $\mathrm{GL}_n(\mathbb{F}_q)$ *and* τ *be an irreducible generic representation of* $\mathrm{GL}_r(\mathbb{F}_q)$, $r < n$. *Then*

$$\gamma(\pi \times \tau,\ \psi) = \sum_{\mathrm{U}_r \backslash \mathrm{GL}_r} \mathcal{B}_{\pi,\psi_n} \begin{pmatrix} 0 & \mathrm{I}_{n-r} \\ m & 0 \end{pmatrix} \mathcal{B}_{\tau,\psi_r^{-1}}(m). \tag{3.1}$$

Green's construction

$\overline{\mathbb{F}_q}$: the algebraic closure of $\mathbb{F}_q$.

$\mathbb{F}_{q^d}$: the unique field extension of $\mathbb{F}_q$ with index d such that $\mathbb{F}_q \subset \mathbb{F}_{q^d} \subset \overline{\mathbb{F}_q}$.

Let η be a character of $\mathbb{F}_{q^n}^\times$ satisfying the following properties:

$$\eta^{q^m - 1} \neq 1, \text{ for all positive } m < n. \tag{4.1}$$

Such a character η is called **regular** and it defines an irreducible cuspidal representation $\pi = \pi_\eta$ of $\mathrm{GL}_n(\mathbb{F}_q)$.

Moreover, any irreducible cuspidal representation corresponds to some regular character.

Regular characters η_1 and η_2 corresponds to the same supercuspidal π if and only if

$$\eta_1(\sigma) = \eta_2(\sigma^{q^t}) \text{ for some } t \geq 1. \tag{4.2}$$

$n \times 1$ gamma factor for $\mathrm{GL}_n(\mathbb{F}_q)$

Let $\mathrm{tr} : \mathbb{F}_{q^n} \mapsto \mathbb{F}_q$ be the reduced trace map and $N : \mathbb{F}_{q^n}{}^{\times} \mapsto \mathbb{F}_{q^n}$ be the reduced norm map.

Theorem

Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, $n \geq 2$ and $\chi \in \widehat{\mathbb{F}}_q^*$. Then

$$\gamma(\pi \times \chi,\ \psi) = (-q^{-1}\chi(-1))^{n-1} \sum_{\sigma \in \mathbb{F}_{q^n}^*} \psi_n(\mathrm{tr}\sigma^{-1})\eta_\pi(\sigma)\chi^{-1}(N(\sigma)),$$

where η_π is the regular character of $\mathbb{F}_{q^n}^*$ corresponding to π in Green's construction.

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The theorem shows gamma factors for $\mathrm{GL}_n(\mathbb{F}_q)$ defined by Roditty coincide with the ones given by Braverman, A. and Kazhdan, D..

4×1 LCT for $\mathrm{GL}_4(\mathbb{F}_3)$

By computing Gauss sums for regular characters of some small rank cases, Zhang, Lei found:

Theorem

Let π_1 and π_2 be irreducible cuspidal representations of $\mathrm{GL}_4(\mathbb{F}_3)$, with the same central character. If

$$\gamma(\pi_1 \times \chi,\ \psi) = \gamma(\pi_2 \times \chi,\ \psi)$$

for all $\chi \in \widehat{\mathbb{F}_3^*}$, then $\pi_1 \cong \pi_2$.

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for all $\chi \in \widehat{\mathbb{F}_3^*}$, then $\pi_1 \cong \pi_2$.

Theorem

Let π_1 and π_2 be irreducible cuspidal representations of $\mathrm{GL}_5(\mathbb{F}_2)$, with the same central character. If

$$\gamma(\pi_1 \times \chi,\ \psi) = \gamma(\pi_2 \times \chi,\ \psi)$$

for all $\chi \in \widehat{\mathbb{F}_2^*}$, then $\pi_1 \cong \pi_2$.

Let $\mathcal{U} \subset \mathcal{M} \subset \mathcal{K}$ be compact open subgroups of $\mathcal{K}$.

Let τ be an irreducible smooth representation of $\mathcal{K}$ and let Ψ be a linear character of $\mathcal{U}$.

Take an open normal subgroup $\mathcal{N}$ of $\mathcal{K}$, which is contained in $\mathrm{Ker}(\tau) \cap \mathcal{U}$.

Let χ_τ be the (trace) character of τ.

Paškūnas, V., Stevens, S. defined
Bessel function $\mathcal{J} : \mathcal{K} \to \mathbb{C}$ of τ by

$$\mathcal{J}(g) := [\mathcal{U} : \mathcal{N}]^{-1} \sum_{u \in \mathcal{U}/\mathcal{N}} \Psi(u^{-1}) \chi_\tau(gu). \tag{5.1}$$

This is independent of the choice of $\mathcal{N}$.

Proposition ([PS08])

Assume that the data introduced above satisfy the following:

- $\tau|_{\mathcal{M}}$ *is an irreducible representation of* $\mathcal{M}$*; and*
- $\tau|_{\mathcal{M}} \cong \mathrm{Ind}_{\mathcal{U}}^{\mathcal{M}}(\Psi)$.

Then the Bessel function $\mathcal{J}$ *of* τ *enjoys the following properties:*

1. $\mathcal{J}(1) = 1$;
2. $\mathcal{J}(hg) = \mathcal{J}(gh) = \Psi(h)\mathcal{J}(g)$ *for all* $h \in \mathcal{U}$ *and* $g \in \mathcal{K}$;
3. *if* $\mathcal{J}(g) \neq 0$, *then* g *intertwines* Ψ; *in particular, if* $m \in \mathcal{M}$, *then* $\mathcal{J}(m) \neq 0$ *if and only if* $m \in \mathcal{U}$;

Let π be an irreducible *unitary* supercuspidal representation of $\mathrm{GL}_n(F)$. By [BH98], there is an extended maximal simple type $(\mathbb{J}, \Lambda)$ in π such that

$$\mathrm{Hom}_{\mathrm{U}_n \cap \mathbb{J}}(\Phi_n, \Lambda) \neq 0.$$

Since Λ restricts to a multiple of some simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi)$, one obtains that $\theta(u) = \Phi_n(u)$ for all $u \in \mathrm{U}_n \cap H^1$. As in [PS08, Definition 4.2], one defines a character $\Psi_n : (J \cap \mathrm{U}_n)H^1 \to \mathbb{C}^\times$ by

$$\Psi_n(uh) := \Phi_n(u)\theta(h), \tag{5.2}$$

for all $u \in J \cap \mathrm{U}_n$ and $h \in H^1$. By [PS08, Theorem 4.4], the data

$$\mathcal{K} = \mathbb{J},\ \tau = \Lambda,\ \mathcal{M} = (J \cap \mathrm{P}_n)J^1,\ \mathcal{U} = (J \cap \mathrm{U}_n)H^1,\ \text{and}\ \Psi = \Psi_n$$

satisfy the conditions in Proposition 5.3 and hence define a Bessel function $\mathcal{J}$.

Define a function $\mathrm{B}_\pi : \mathrm{GL}_n(F) \to \mathbb{C}$ by

$$\mathrm{B}_\pi(g) := \begin{cases} \Phi_n(u)\mathcal{J}(j) & \text{if } g = uj \text{ with } u \in \mathrm{U}_n,\ j \in \mathbb{J}, \\ 0 & \text{otherwise,} \end{cases} \tag{5.3}$$

which is well-defined by Proposition 5.3(ii). Then, by [PS08], B_π is a Whittaker function for π. By Proposition 5.3, the restriction of B_π to P_n has a particularly simple description: for $g \in \mathrm{P}_n$,

$$\mathrm{B}_\pi(g) = \begin{cases} \Psi_n(g) & \text{if } g \in (J \cap \mathrm{U}_n)H^1; \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Gamma Factors for Level Zero Cuspidals

In the level zero case, $J = \mathrm{K}_n, \mathbb{J} = \mathrm{Z}_n\mathrm{K}_n, J^1 = H^1 = \mathfrak{P}_n, \theta = 1$, and $\mathcal{N} = \mathfrak{P}_n$ and

$$\omega\tau_i = \Lambda,\ \mathcal{M} = (\mathrm{K}_n \cap \mathrm{P}_n)\mathfrak{P}_n,\ \mathcal{U} = (\mathrm{K}_n \cap \mathrm{U}_n)\mathfrak{P}_n.$$

$\pi_i \cong \mathrm{c} - \mathrm{Ind}_{\mathrm{Z}_n(F)K_n(F)}^{\mathrm{GL}_n(F)} \omega\tau_i$, for $i = 1,\ 2$, where τ_1(resp. τ_2) is an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$(resp. $\mathrm{GL}_t(\mathbb{F}_q)$), and q is the cardinality of the residue field of F.

Theorem

For $n \geq 2$, let π_1 (resp. π_2) be an irreducible, level zero supercuspidal, unitary representations of $\mathrm{GL}_n(F)$(resp. $\mathrm{GL}_t(F)$), $t < n$. Then

$$\omega_{\pi_2}(-1)^{n-1}\Gamma(s, \pi_1 \times \pi_2, \psi_F) = q^{\frac{t(n-t-1)}{2}}\gamma(\tau_1 \times \tau_2, \psi).$$

By Corollary 2.4, π and π_2 possess the same central character ω.

By the construction of supercuspidal representation of $\mathrm{GL}_n(F)$, we may assume that

$$\pi_i \cong \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}_n(F)K_n(F)}^{\mathrm{GL}_n(F)}\omega\tau_i,$$

for $i=1,\ 2,$ where τ_1(resp. τ_2) is an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$(resp. $\mathrm{GL}_t(\mathbb{F}_q)$), and q is the cardinality of the residue field of F.

Since $\mathcal{U}/\mathcal{N} \cong \mathrm{U}_n(\mathbb{F}_q)$, the Bessel function for τ_i is given by

$$\begin{aligned}\mathcal{J}_i(g) &= [\mathcal{U}:\mathcal{N}]^{-1} \sum_{u \in \mathcal{U}/\mathcal{N}} \Psi(u^{-1})\chi_{\tau_i}(gu) \\ &= \frac{1}{|\mathrm{U}_n(\mathbb{F}_q)|} \sum_{u \in \mathcal{U}/\mathcal{N}} \Psi(u^{-1})\chi_{\tau_i}(gu), \text{ for } i = 1,\ 2.\end{aligned}$$

The Whittaker function B_{π_i}, defined in Eq. (5.3), are supported in $\mathrm{U}_n\mathrm{K}_n\mathrm{Z}_n$. For element $g \in \mathrm{K}_n$, define by $\bar{g} \in \mathbb{F}_q$ its reduction modulo $\mathfrak{p}$. Note that for $u \in \mathrm{U}_n \cap \mathrm{K}_n,\ k \in \mathrm{K}_n,\ \mathcal{J}_i(ku) = \Phi_n(u)\mathcal{J}_i(k)$, so

$$\mathcal{J}_i(k) = \mathcal{B}_{\tau_i}(\bar{k}), \text{ for } k \in \mathrm{K}_n, \tag{6.1}$$

where $\mathcal{B}_{\tau_i}$ is the Bessel function of τ_i in terms of ψ_n. Then ψ_n is the standard Whittaker character for $\mathrm{U}_n(\mathbb{F}_q)$.

4×1 and 5×1 LCT for Level Zero cuspidals

Theorem

Let π be an an irreducible, level zero supercuspidal, unitary representations of $\mathrm{GL}_4(F)$, where the residue field of F has 3 elements. Then π is uniquely determined by the set of twisted gamma factors $\{\Gamma(s, \pi_i \times \chi, \psi_F) \mid \chi \in \widehat{F}^{\times}\}$.

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Theorem

Let π be an an irreducible, level zero supercuspidal, unitary representations of $\mathrm{GL}_5(F)$, where the residue field of F has 2 elements. Then π is uniquely determined by the set of twisted gamma factors $\{\Gamma(s, \pi_i \times \chi, \psi_F) \mid \chi \in \widehat{F}^\times\}$.

Distinction and Gamma factors

Let E be a quadratic extension of a p-adic field F such that the characteristic of the residue field of F is odd. Let ψ_E be a non trivial character of E such that its restriction to F is trivial.

Definition

A representation (π, V) *of* $\mathrm{GL}_n(E)$ *is called* $\mathrm{GL}_n(F)$*-distinguished if there exists a non-zero linear form*

$$T : V \mapsto \mathbb{C} \text{ such that } T(\pi(g)v) = T(v),\ v \in V,\ g \in \mathrm{GL}_n(F).$$

Let $H_n = \mathrm{GL}_n(F)$.

$\mathrm{GL}_2(F)$

Hakim studied distinction with special values of Gamma factor for $\mathrm{GL}_2(F)$ and Ok generalized his result to supercuspidal representations of $\mathrm{GL}_n(F)$.

Theorem ([Ha91])

Let π be an irreducible unitary generic representation of $\mathrm{GL}_2(E)$ with central character whose restriction to F is trivial. Then

$$\Gamma(\frac{1}{2}, \pi \times \tau, \psi_E) = 1$$

for all $F^\times$-distinguished character τ of $E^\times$, if and only if π is H_2-distinguished.

$n \times (n-1)$ distinction

Theorem ([Ok97])

Let π be an irreducible, unitary, supercuspidal representation of $\mathrm{GL}_n(E)$ with central character whose restriction to F is trivial. Then the following are equivalent:

1. *π is H_n-distinguished.*
2. $$\Gamma(\frac{1}{2}, \pi \times \tau, \psi_E) = 1$$
   for any $\mathrm{GL}_{n-1}(F)$-distinguished unitary generic representation τ of $\mathrm{GL}_{n-1}(E)$.

Distinction and Special Value of Gamma Factors

Theorem ([Of11])

Let π (resp. τ) be a smooth, irreducible generic and H_n-distinguished (resp. $\mathrm{GL}_t(F)$-distinguished) representation of $\mathrm{GL}_n(E)$ (resp. $\mathrm{GL}_t(E)$. Then

$$\epsilon(\frac{1}{2}, \pi \times \tau, \psi_E) = \Gamma(\frac{1}{2}, \pi \times \tau, \psi_E) = 1.$$

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Theorem ($n \times (n-2)$-distinction, [HO15])

Let π be a supercuspidal representation of $\mathrm{GL}_n(E)$, $n \geq 3$. Then π is H_n-distinguished if and only if

$$\Gamma(\frac{1}{2}, \pi \times \tau, \psi_E) = 1$$

for all irreducible generic H_r-distinguished representations τ of $\mathrm{GL}_r(E)$, for $r = 1, \cdots, n-2$.

The End!!

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