The character and the wave front set correspondence in the stable range

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Howe's Theory of Rank

$$E = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.$$

$$\begin{aligned}
\mathrm{Sp}_{2n} &= \left\{ g \in \mathrm{M}_{2n}(\mathbb{R});\ g^t \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} g = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \right\} \\
\mathrm{P} &= \left\{ \begin{pmatrix} a & c \\ 0 & {}^t a^{-1} \end{pmatrix};\ a \in \mathrm{GL}_n,\ c = {}^t c \in \mathrm{M}_n \right\} \\
\mathrm{N} &= \left\{ \begin{pmatrix} I & c \\ 0 & I \end{pmatrix};\ c = {}^t c \in \mathrm{M}_n \right\} \\
\hat{\mathrm{N}} &= \{ \hat{c} = {}^t \hat{c} \in \mathrm{M}_n \}, \quad \hat{c}\left(\begin{pmatrix} I & c \\ 0 & I \end{pmatrix} \right) = \exp(2\pi i \, \mathrm{tr}(\hat{c} c)).
\end{aligned}$$

Suppose

Π is an irreducible unitary representation of a cover of Sp_{2n}
and
$\Pi|_{\mathrm{N}}$ is supported on matrices of maximal rank $r < n$.

Then

Π factors through a double cover

$$\widetilde{\mathrm{Sp}}_{2n} \to \mathrm{Sp}_{2n}$$

of the symplectic group, which is trivial if r is even and is equal to the metaplectic cover if r is odd. Moreover...

$$\begin{aligned} \mathrm{P}_1 &= \left\{ \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_2 & b_1^* \\ 0 & 0 & {}^t a_1^{-1} \end{pmatrix} \in \mathrm{Sp}_{2n};\ a_1 \in \mathrm{GL}_r, a_2 \in \mathrm{Sp}_{2n-2r} \right\} \\ &= (\mathrm{GL}_r \times \mathrm{Sp}_{2n-2r}) \cdot \mathrm{N}_1 \end{aligned}$$

There are unique integers $p \geq q \geq 0$ with $p + q = r$, an irreducible unitary representation Π'_O of $\widetilde{\mathrm{O}}_{p,q}$ and a character $\alpha : \widetilde{\mathrm{O}}_{p,q} \to \mathbb{C}$ such that

$$\Pi|_{\widetilde{\mathrm{P}}_1} = \mathrm{Ind}_{(\mathrm{O}_{p,q}\times\mathrm{Sp}_{2n-2r})^{\sim}\cdot\mathrm{N}_1}^{\widetilde{\mathrm{P}}_1}((\alpha\Pi'_O \otimes 1) \otimes \omega_{p,q}) \,.$$

There is a subset $\left(\widehat{\widetilde{\mathrm{Sp}}_{2n}}\right)_{p,q} \subseteq \widehat{\widetilde{\mathrm{Sp}}_{2n}}$ such that the above formula gives a bijection

$$\left(\widehat{\widetilde{\mathrm{Sp}}_{2n}}\right)_{p,q} \ni \Pi \longleftrightarrow \Pi'_O \in \hat{\mathrm{O}}_{p,q} \,.$$

E.g. the rank one representations ($r = 1$) are the two Weil representations of the metaplectic group.

The Metaplectic Group

$(\mathsf{W}, \langle\cdot,\cdot\rangle)$; $\mathrm{Sp}, \mathfrak{sp} \subseteq \mathrm{End}(\mathsf{W})$; $J \in \mathfrak{sp}$, $J^2 = -I$, $\langle J\cdot,\cdot\rangle > 0$; $g, g_1, g_2 \in \mathrm{Sp}$;
$\chi(r) = e^{2\pi i r}$, $r \in \mathbb{R}$.
The restriction of $J_g = J^{-1}(g-1)$ to $J_g\mathsf{W}$ is invertible. Let

$$C(g_1, g_2) = \sqrt{\left|\frac{\det(J_{g_1})_{J_{g_1}\mathsf{W}} \det(J_{g_2})_{J_{g_2}\mathsf{W}}}{\det(J_{g_1 g_2})_{J_{g_1 g_2}\mathsf{W}}}\right|}\chi(\frac{1}{8}\mathrm{sgn}(q_{g_1,g_2})),$$

where $\mathrm{sgn}(q_{g_1,g_2})$ is the signature of the symmetric form

$$\begin{aligned} q_{g_1,g_2}(u', u'') &= \frac{1}{2}\langle(g_1+1)(g_1-1)^{-1}u', u''\rangle \\ &+ \frac{1}{2}\langle(g_2+1)(g_2-1)^{-1}u', u''\rangle \\ &\quad (u', u'' \in (g_1-1)\mathsf{W} \cap (g_2-1)\mathsf{W}). \end{aligned}$$

$$\widetilde{\mathrm{Sp}} = \left\{\tilde{g} = (g, \xi) \in \mathrm{Sp} \times \mathbb{C}, \quad \xi^2 = i^{\dim(g-1)\mathsf{W}} \det(J_g)^{-1}_{J_g\mathsf{W}}\right\},$$

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 C(g_1, g_2)).$$

The Weil Representation

$\mathsf{W} = \mathrm{X} \oplus \mathrm{Y}$; Op : $\mathcal{S}^*(\mathrm{X} \times \mathrm{X}) \to \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X}))$

$$\mathrm{Op}(K)v(x) = \int_{\mathrm{X}} K(x, x')v(x')\, dx'.$$

Weyl transform $\mathcal{K} : \mathcal{S}^*(\mathsf{W}) \to \mathcal{S}^*(\mathrm{X} \times \mathrm{X})$

$$\mathcal{K}(f)(x, x') = \int_{\mathrm{Y}} f(x - x' + y)\chi(\frac{1}{2}\langle y, x + x'\rangle)\, dy.$$

A singular imaginary Gaussian

$$\chi_{c(g)}(u) = \chi(\frac{1}{4}\langle (g+1)(g-1)^{-1}u, u\rangle) \qquad (u = (g-1)w,\ w \in \mathsf{W}).$$

For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}$ define

$$\Theta(\tilde{g}) = \xi, \qquad T(\tilde{g}) = \Theta(\tilde{g})\chi_{c(g)}\mu_{(g-1)\mathsf{W}}, \qquad \omega(\tilde{g}) = \mathrm{Op} \circ \mathcal{K} \circ T(\tilde{g}),$$

where $\mu_{(g-1)\mathsf{W}}$ is the Lebesgue measure on the subspace $(g-1)\mathsf{W}$ normalized so that the volume of the unit cube with respect to the form $\langle J\cdot, \cdot\rangle$ is 1. In these terms, $(\omega, \mathrm{L}^2(\mathrm{X}))$ is the Weil representation of $\widetilde{\mathrm{Sp}}$ attached to the character χ.

Dual Pairs

Subgroups $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp}(\mathsf{W})$ act reductively on W, G' is the centralizer of G in Sp and G is the centralizer of G' in Sp.

G, G'	stable range
$\mathrm{GL}_n(\mathbb{D}),\ \mathrm{GL}_m(\mathbb{D})$	$n \geq 2m$
$\mathrm{O}_{p,q},\ \mathrm{Sp}_{2n}(\mathbb{R})$	$p, q \geq 2n$
$\mathrm{Sp}_{2n}(\mathbb{R}),\ \mathrm{O}_{p,q}$	$n \geq p+q$
$\mathrm{O}_p(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$	$p \geq 4n$
$\mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}_p(\mathbb{C})$	$n \geq p$
$\mathrm{U}_{p,q},\ \mathrm{U}_{r,s}$	$p, q \geq r+s$
$\mathrm{Sp}_{p,q},\ \mathrm{O}^*_{2n}$	$p, q \geq n$
$\mathrm{O}^*_{2n},\ \mathrm{Sp}_{p,q}$	$n \geq 2(p+q)$

Howe's Correspondence

$\mathcal{R}(\widetilde{\mathrm{G}},\omega) \subseteq \mathcal{R}(\widetilde{\mathrm{G}})$ irreducible admissible representations realized as quotients of $\mathcal{S}(X)$ by closed $\widetilde{\mathrm{G}}$-invariant subspaces. $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}},\omega)$, $N_\Pi \subseteq \mathcal{S}(X)$ the intersection of all the closed G-invariant subspaces $N \subseteq \mathcal{S}(X)$ such that Π is infinitesimally equivalent to $\mathcal{S}(X)/N$. This is a representation of both $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$. It is infinitesimally isomorphic to

$$\Pi \otimes \Pi_1',$$

for some representation Π_1' of $\widetilde{\mathrm{G}}'$. Howe proved that Π_1' is a finitely generated admissible quasisimple representation of $\widetilde{\mathrm{G}}'$, which has a unique irreducible quotient $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}',\omega)$. Conversely, starting with $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}',\omega)$ and applying the above procedure with the roles of G and G' reversed, we arrive at the representation $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}},\omega)$. The resulting bijection

$$\mathcal{R}(\widetilde{\mathrm{G}},\omega) \ni \Pi \longleftrightarrow \Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}',\omega)$$

is called Howe's correspondence, or local θ correspondence, for the pair G, G'.

Jian-Shu Li

Let $(\mathrm{G}, \mathrm{G}')$ be a dual pair in the stable range with G' - the smaller member. If Π' is unitary then Π is unitary. Howe's theory of rank is compatible with Howe's correspondence. In particular for the pair $(\mathrm{Sp}_{2n}(\mathbb{R}), \mathrm{O}_{p,q})$ with $p + q < n$, there is a character α' of the preimage of the orthogonal group in the metaplectic group such that

$$\Pi' = \alpha' \Pi'_O .$$

In particular $\Pi|_{\widetilde{\mathrm{P}}_1}$ is understood in terms of Π'.

An explicit description of all the $\Pi(g)$, $g \in \widetilde{\mathrm{G}}$, seems out of reach. Instead one may try to describe the distribution character Θ_Π in terms of $\Theta_{\Pi'}$.

The wave front set of a distribution

The wave front set of a distribution u on an Euclidean space V at a point $v \in V$, denoted $WF_v(u)$ is the complement of the set of all pairs (v, v^*), $v^* \in V^*$, such that there is a test function $\phi \in C_c^\infty(V)$ with $\phi(v) \neq 0$ and an open cone $\Gamma \subseteq V^*$ containing v^* such that

$$|\mathcal{F}(\phi u)(v_1^*)| \leq C_N(1 + |v_1^*|)^{-N} \qquad (v_1^* \in \Gamma,\ N = 0, 1, 2, ...)\,.$$

This notion behaves well under diffeomorphisms, so one may define $WF(u) \subseteq T^*M$ as the union of the wave front sets at the individual points for any distribution u on a manifold M.

The Cauchy Harish-Chandra Integral

Assume that the rank of G' is smaller or equal than the rank of G.
θ-invariant Cartan subgroup $\mathrm{H}' \subseteq \mathrm{G}'$; $\mathrm{A}' = \{h \in \mathrm{H}';\ \theta(h) = h^{-1}\}$;
$\mathrm{A}'' \subseteq \mathrm{Sp}$ centralizer of A', $\mathrm{A}''' \subseteq \mathrm{Sp}$ centralizer of A''; $d\dot{w}$ measure on $\mathrm{A}'''\backslash \mathsf{W}$ defined by

$$\int_{\mathsf{W}} \phi(w)\, dw = \int_{\mathrm{A}'''\backslash \mathsf{W}} \int_{\mathrm{A}'''} \phi(aw)\, da\, d\dot{w}.$$

$$\mathrm{Chc}(f) = \int_{\mathrm{A}'''\backslash \mathsf{W}} \int_{\widetilde{\mathrm{A}''}} f(g) T(g)(w)\, dg\, d\dot{w} \qquad (f \in C_c^\infty(\widetilde{\mathrm{A}''})).$$

For any $h' \in \mathrm{H}'^{reg}$, the intersection of the wave front set of the distribution (**??**) with the conormal bundle of the embedding

$$\widetilde{\mathrm{G}} \ni \widetilde{g} \longrightarrow \widetilde{h'}\widetilde{g} \in \widetilde{\mathrm{A}''}$$

is empty. Hence there is a unique restriction of the distribution Chc to $\widetilde{\mathrm{G}}$, denoted $\mathrm{Chc}_{\widetilde{h'}}$.

The distribution $\Theta'_{\Pi'}$

Set $D(h) = |\det(Ad(h^{-1}) - 1)_{\mathfrak{g}'/\mathfrak{h}'}|$.

Weyl integration formula

$$\int_{\widetilde{\mathrm{G}}'} \phi(g)\, dg = \sum \frac{1}{|W(\mathrm{H}')|} \int_{\widetilde{\mathrm{H}'^{reg}}} D(h) \int_{\widetilde{\mathrm{G}}'/\widetilde{\mathrm{H}}'} \phi(g\widetilde{h}g^{-1})\, d\dot{g}\, d\widetilde{h},$$

Define

$$\Theta'_{\Pi'}(f) = C_{\Pi'} \sum \frac{1}{|W(\mathrm{H}')|} \int_{\widetilde{\mathrm{H}'^{reg}}} D(h)\, \Theta_{\Pi'}(\widetilde{h}^{-1})\mathrm{Chc}_{\widetilde{h}}(f)\, d\widetilde{h}.$$

This is an invariant eigen-distribution on $\widetilde{\mathrm{G}}$ with the correct infinitesimal character.

The equality $\Theta'_{\Pi'} = \Theta_\Pi$

Theorem

Let $(\mathrm{G}, \mathrm{G}')$ be a dual pair in the stable range with G' - the smaller member. If Π' is unitary then $\Theta'_{\Pi'} = \Theta_\Pi$.

Idea of the proof. We show that the two distributions are equal on a Zariski open subset $\widetilde{\mathrm{G}}'' \subseteq \widetilde{\mathrm{G}}$. Since both Θ_Π and $\Theta'_{\Pi'}$ is an invariant eigendistribution, Harish-Chandra Regularity Theorem implies that they are equal everywhere.

The moment maps

The maps $\tau_{\mathfrak{g}} : \mathrm{W} \to \mathfrak{g}^*$ and $\tau_{\mathfrak{g}'} : \mathrm{W} \to \mathfrak{g}'^*$ defined by

$$\tau_{\mathfrak{g}}(z) = \langle z(w), w\rangle \qquad (z \in \mathfrak{g},\ w \in \mathrm{W})$$

(and similarly for $\mathfrak{g}'$) occur naturally in the structure of the Weil representation.

If $\mathcal{N}(\mathfrak{g}^*) \subseteq \mathfrak{g}^*$ denotes the subset of the nilpotent elements, then

$$\tau_{\mathfrak{g}}(\tau_{\mathfrak{g}'}^{-1}(\mathcal{N}(\mathfrak{g}'^*)) \subseteq \mathcal{N}(\mathfrak{g}^*).$$

The equality $WF(\Pi) = \tau_{\mathfrak{g}}(\tau_{\mathfrak{g}'}^{-1}(WF(\Pi')))$

$WF(\Pi) = WF_1(\Theta_\Pi) \subseteq \mathcal{N}(\mathfrak{g}^*)$.

Theorem

Let $(\mathrm{G}, \mathrm{G}')$ be a dual pair in the stable range with G' - the smaller member. If Π' is unitary then $WF(\Pi) = \tau_{\mathfrak{g}}(\tau_{\mathfrak{g}'}^{-1}(WF(\Pi')))$.

This follows from the equality $\Theta'_{\Pi'} = \Theta_\Pi$ and the properties of the distribution $\Theta'_{\Pi'}$ except when $\mathrm{G}' \neq \mathrm{G}'_1$ (the Zariski identity component of G'), the restriction of Π' to $\widetilde{\mathrm{G}}'_1$ is irreducible and $WF(\Pi')$ contains more than one orbit of maximal dimension. In that last case, we use the result of Loke and Ma, who computed the correspondence of the associated varieties of the Harish-Chandra modules of Π and Π' and a theorem of Schmid and Vilonen proving a conjecture of Barbasch and Vogan.

Remark: another posssibility would be to extend the notion of the Cauchy Harish-Chandra integral to the full group $\widetilde{\mathrm{G}}'$, not just $\widetilde{\mathrm{G}}'_1$.

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Thank you