Some small genuine representations of a nonlinear double cover and the model orbit

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Outline

$G_{\mathbb{C}}$: simply connected, semisimple, simply laced, complex Lie group
G: real form of $G_{\mathbb{C}}$ with $\pi_1(G) \neq 1$
$\widetilde{G}$: nonlinear two-fold cover of G

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- Characterize Lift($\mathbb{C}$) explicitly when G is split
- Demonstrate that the K-types of Lift($\mathbb{C}$) match up with the K-structure of the regular functions on a model orbit of type D_{2m}

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Some other examples which are more well-known (but not in the setting):

- $G = GL(n, \mathbb{R})$
- $G = Sp(2n, \mathbb{R})$, $\widetilde{G} = Mp(2n, \mathbb{R})$, the metaplectic group

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Identify the kernel of the covering map $p : \widetilde{G} \to G$ with $\{1, \epsilon\}$.

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Key Tool: Lifting (Flicker-Kazhdan-Patterson)
We expect Lifting is an operator taking representations of G to genuine representations of $\widetilde{G}$. Write $\mathrm{Lift}_G^{\widetilde{G}}(\pi)$.

Lifting

Definition

Let π be an admissible representation of G, Θ_π be the global character of π, (which is regarded as a function on G', set of regular s.s. elts of G). For $\widetilde{g} \in \widetilde{G'}$, define

$$\mathrm{Lift}_G^{\widetilde{G}}(\Theta_\pi)(\widetilde{g}) = \sum_{\{h \in G \,|\, h^2 = p(\widetilde{g})\}} \Delta(h, \widetilde{g})\Theta_\pi(h),$$

where $\Delta(h, \widetilde{g})$ is called a transfer factor satisfying certain conditions.

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Theorem (Adams-Herb)

Assume the setting in the beginning. Then there exists definition for $\Delta(h, \widetilde{g})$ such that if π is a stable representation of G, then $\mathrm{Lift}_G^{\widetilde{G}}(\pi)$ is 0 or a genuine virtual representation of $\widetilde{G}$.

Lifting

Write $\mathrm{Lift}_{G}^{\widetilde{G}}(\pi) = \sum_{\widetilde{\pi}} a_{\widetilde{\pi}} \widetilde{\pi}$, where $\widetilde{\pi}$ is irr., genuine, $a_{\widetilde{\pi}} \in \mathbb{Z}$.

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Example (Theorem by Adams-Huang)

Let $G = GL(n, \mathbb{R})$. Then

(1) If π is an irr. unitary representation of G, then $\mathrm{Lift}(\pi)$ is irr. unitary or 0.

(2)

$$\mathrm{Lift}(\mathbb{C}) = \begin{cases} T_n & \text{if } n \text{ is even} \\ T_n(\chi) & \text{if } n \text{ is odd} \end{cases}$$

$$\mathrm{Lift}(sgn) = \begin{cases} Speh(1/2) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Lifting

Write $\mathrm{Lift}_G^{\widetilde{G}}(\pi) = \sum_{\widetilde{\pi}} a_{\widetilde{\pi}} \widetilde{\pi}$, where $\widetilde{\pi}$ is irr., genuine, $a_{\widetilde{\pi}} \in \mathbb{Z}$.

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$\star$ T_n is the irreducible quotient of the minimal principal series, with the Pin representation as its lowest K-type. (See Huang's thesis for a full description for T_n.)

Goal: Describe Lift($\mathbb{C}$) for various groups (as in the setting)

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Goal: Describe Lift($\mathbb{C}$) for various groups (as in the setting)
(In such setting, G is connected and hence has only one one-dimensional representation.)

Describe Lift($\mathbb{C}$) as a set of genuine small representations

Definition

Define $\prod^{s}_{\rho/2}(\widetilde{G}) = \{\widetilde{\pi} \,|\, \widetilde{\pi}$ is gen irr. with inf. char. $\rho/2$ and maximal τ-invariant$\}$

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Lemma

There exists a unique complex nilpotent orbit $\mathcal{O}$ such that $AV(I_{\widetilde{\pi}}) = \overline{\mathcal{O}}$, $\forall \widetilde{\pi} \in \prod^s_{\rho/2}(\widetilde{G})$. This $\mathcal{O}$ can be computed explicitly.

Notation:
$AV(I_{\widetilde{\pi}}) = \overline{O}$ is the (complex) associated variety of $\widetilde{\pi}$.

Type	A_{n-1} ($\mathfrak{g} = \mathfrak{sl}_n$)		D_n ($\mathfrak{g} = \mathfrak{so}_{2n}$)		E_6	E_7	E_8
n	$2m$	$2m+1$	$2m$	$2m+1$			
$\mathcal{O}$	$[2^m]$	$[2^m\,1]$	$[3\,2^{n-2}\,1]$	$[3\,2^{n-3}\,1^3]$	$3A_1$	$4A_1$	$4A_1$

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where $\mathcal{O}$ is as in the previous Lemma and table.

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⋆ The first containment is the generalization of the case of $GL(n, \mathbb{R})$ by Adams-Huang.

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Notation:
$AV(I_{\widetilde{\pi}}) = \overline{\mathcal{O}}$ is the (complex) associated variety of $\widetilde{\pi}$.
$AV(\widetilde{\pi}) = \overline{\mathcal{O}_i}$ is the (real) associated variety of $\widetilde{\pi}$ (in our case), where $\mathcal{O}_i$ is a real form of $\mathcal{O}$.

Type A_{n-1}, $\mathfrak{g} = \mathfrak{sl}_n$

n	G	$\{\mathcal{O}_i\}$	$\#\{\mathcal{O}_i\}$
$2m$	$SL(n, \mathbb{R})$ (split)	$\begin{array}{\|c\|c\|}\hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$ (I, II)	2
$2m+1$	$SL(n, \mathbb{R})$ (split)	$\begin{array}{\|c\|c\|}\hline & \\ \hline & \\ \hline & \\ \hline \multicolumn{1}{\|c\|}{} & \\ \cline{1-1} \end{array}$	1
$2m$	$SU(m, m)$(quasisplit)	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline + & - \\ \hline \end{array}$	$m+1$
$2m+1$	$SU(m+1, m)$ (quasisplit)	$\begin{array}{\|c\|c\|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline \multicolumn{1}{\|c\|}{\pm} & \\ \cline{1-1} \end{array}$	$m+1$

Type D_n, $\mathfrak{g} = \mathfrak{so}_{2n}$

n	G	$\{\mathcal{O}_i\}$	$\#\{\mathcal{O}_i\}$
$2m$	$Spin(n, n)$ (split)	$\begin{array}{\|c\|c\|c\|} \pm & \mp & \pm \\ + & - & \\ + & - & \\ \mp & & \end{array}$ (I, II)	4
$2m+1$	$Spin(n, n)$ (split)	$\begin{array}{\|c\|c\|c\|} \pm & \mp & \pm \\ + & - & \\ + & - & \\ \pm & & \\ \mp & & \\ \mp & & \end{array}$	2
$2m$	$Spin(n+1, n-1)$ (quasisplit)	$\begin{array}{\|c\|c\|c\|} + & - & + \\ + & - & \\ + & - & \\ + & & \end{array}$	1

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$2m+1$	$Spin(n+2, n-2)$	$\begin{array}{lll} + & - & + \\ + & - & \\ + & - & \\ + & & \\ + & & \\ + & & \end{array}$	1

Type E

Type	G	$\{\mathcal{O}_i\}$	$\#\{\mathcal{O}_i\}$
E_6	$E_6(A_1 \times A_5)$ (quasisplit)	$\#4, 5$	2
E_6	$E_6(C_4)$ (split)	$\#3$	1
E_7	$E_7(A_7)$ (split)	$\#8, 9$	2
E_8	$E_8(D_8)$ (split)	$\#6$	1

Remark

The groups listed in these tables are the only real simply laced groups such that $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}} \neq \phi$. (Most of them are quasisplit).

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Corollary

For G not listed in the above tables, $\mathrm{Lift}(\mathbb{C}) = 0$.

Construct Lift($\mathbb{C}$) for G split

- From a paper of Adams-Barbasch-Paul-Trapa-Vogan, there defines a set of genuine irreducible representations $\{Sh_i\}$, called Shimura representations, which are the irreducible quotients of genuine minimal principal series.

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Type	G	$\widetilde{K}$	$\#\{Sh_i\}$	LKT of Sh_i
A_{2m-1}	$SL(2m,\mathbb{R})$	$Spin(2m)$	2	$Spin_\pm$
A_{2m}	$SL(2m+1,\mathbb{R})$	$Spin(2m+1)$	1	$Spin$
D_{2m}	$Spin(2m,2m)$	$Spin(2m)\times Spin(2m)$	4	$1\otimes Spin_\pm$, $Spin_\pm\otimes 1$
D_{2m+1}	$Spin(2m+1,2m+1)$	$Spin(2m+1)\times Spin(2m+1)$	2	$1\otimes Spin$, $Spin\otimes 1$
E_6	$E_6(C_4)$	$Sp(8)$	1	$\mathbb{C}^8$
E_7	$E_7(A_7)$	$SU(8)$	2	$\mathbb{C}^8, (\mathbb{C}^8)^*$
E_8	$E_8(D_8)$	$Spin(16)$	1	$\mathbb{C}^{16}$

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Theorem

$\prod_{R_D}(\widetilde{G}) \subseteq \prod^s_{\rho/2}(\widetilde{G})$ and $\prod_{R_D}(\widetilde{G}) \leftrightarrow \{(\chi_i, \mathcal{O}_j)\}$
(The representation we constructed are small with inf. char. $\rho/2$.)

Example

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$\mathbf{1}_\pm = (1, \cdots, \pm 1)$

D_4

	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$
χ_1	Sh_1 $Spin_+ \boxtimes \mathbb{C}$	π_1 $\Gamma_- \boxtimes \mathbb{C}$	δ_1 $Spin_+ \boxtimes \mathbf{1}_+$	τ_1 $Spin_+ \boxtimes \mathbf{1}_-$
χ_2	π_2 $\Gamma_+ \boxtimes \mathbb{C}$	Sh_2 $Spin_- \boxtimes \mathbb{C}$	τ_2 $Spin_- \boxtimes \mathbf{1}_+$	δ_2 $Spin_- \boxtimes \mathbf{1}_-$
χ_3	δ_3 $\mathbf{1}_+ \boxtimes Spin_+$	τ_3 $\mathbf{1}_- \boxtimes Spin_+$	Sh_3 $\mathbb{C} \boxtimes Spin_+$	π_3 $\mathbb{C} \boxtimes \Gamma_-$
χ_4	τ_4 $\mathbf{1}_+ \boxtimes Spin_-$	δ_4 $\mathbf{1}_- \boxtimes Spin_-$	π_4 $\mathbb{C} \boxtimes \Gamma_+$	Sh_4 $\mathbb{C} \boxtimes Spin_-$

Main Theorem

For G split, we have
$\prod_{R_D}(\widetilde{G}) = \mathrm{Lift}(\mathbb{C}) = \prod^s_{\rho/2}(\widetilde{G}) = \prod^{\mathcal{O}}_{\rho/2}(\widetilde{G}) \leftrightarrow \{(\chi_i, \mathcal{O}_j)\}$

Sketch of proof. Recall that $\mathrm{Lift}(\mathbb{C}) \subseteq \prod^s_{\rho/2}(\widetilde{G}) = \prod^{\mathcal{O}}_{\rho/2}(\widetilde{G})$ for G not necessarily split.
The fact that $\prod_{R_D}(\widetilde{G}) = \prod^s_{\rho/2}(\widetilde{G})$ is obtained by counting the elements in $\prod^s_{\rho/2}(\widetilde{G})$ by a Weyl group calculation.

Work in progress:

- Generalize the results in the paper of Adams-Barbasch-Paul-Trapa-Vogan to the quasisplit groups (i.e. $G = SU(m,m), SU(m+1,m), Spin(n+1,n-1), E_6(A_1 \times A_5))$
- Study the structure of the double cover $\widetilde{G}$ of these groups
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The conjecture is proved for split type A_{n-1} by Lucas in his thesis.

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General setting:

G: a complex, semisimple algebraic group, $\mathfrak{g}$ =Lie(G),

K: max. cpt subgroup, $e \in \mathfrak{g}$: nilp. element

$\mathcal{O} := G \cdot e$

$R(\mathcal{O}) :=$ ring of regular functions on $\mathcal{O}$

$\star$ Fact. $R(\mathcal{O})$ carries a K-representation.

Two stages

(A) Decompose $R(\mathcal{O}) = \sum_{\lambda \in K^\wedge} V(\lambda)$, where $V(\lambda)$ is an irreducible representation of K with highest weight λ.
(B) Compute the K-types of π where $\pi \in G^\wedge$ with $AV(\pi) = \overline{\mathcal{O}}$.

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There are some orbits that we are mainly interested in.

Definition

A nilpotent orbit $\mathcal{O}$ is called a model orbit if all K-multiplicities in $R(\mathcal{O})$ are 0 or 1.

- For a complex s.c., s.s Lie group G, McGovern points out that $\mathcal{O}$ is a model orbit for $\mathcal{O}$ in the above table (in the simply laced cases), but in his paper he did not calculate the decomposition of $R(\mathcal{O})$ explicitly.

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Goal: Generalize this problem to some cases when G is real (joint work with Barbasch).

Type D_n, $n = 2m$

$\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, $\mathcal{O} = [3\ 2 \cdots 2\ 1]$, $\widetilde{G} = \widetilde{Spin}(n, n)$,
$\widetilde{K} = Spin(n) \times Spin(n)$.
$\mathcal{O} \cap \mathfrak{g}_0 = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_4$. (Each orbit $\mathcal{O}_i$ has 4 representations attached to it.)
Take $\mathcal{O}_1$, say, $\mathcal{O}_1 = \widetilde{G} \cdot e$, where
$e = X(e_1 + e_n) + X(e_1 - e_n) + X(e_2 - e_{n+1}) + \cdots X(e_m - e_{n-1})$.
We calculate $R(\widetilde{G} \cdot e, \chi)|_{\widetilde{K}} = Ind^{\mathfrak{k}}_{C_{\mathfrak{k}}(e)}(\chi)$, where χ is an algebraic character of $C_{\mathfrak{k}}(e)$.
Let $V(\lambda)$ be a genuine $\widetilde{K}$-type, i.e. $\lambda = (a_1, \ldots, a_m \mid b_1, \ldots, b_m)$, $a_1 \geq \ldots |a_m|$, $b_1 \geq \cdots \geq |b_m|$ and $a_i - b_j \in \mathbb{Z} + \frac{1}{2}$. It turns out that

$$[V(\lambda) : R(\mathcal{O}_1)|_{\widetilde{K}}] \neq 0 \iff a_1 \geq b_1 + \frac{1}{2} \geq a_2 \geq b_2 + \frac{1}{2} \geq \cdots \geq |b_m| + \frac{1}{2}$$

and

$$R(\mathcal{O}_1, \frac{1}{2}) = Sh_1|_{\widetilde{K}} + \pi_2|_{\widetilde{K}} + \delta_3|_{\widetilde{K}} + \tau_4|_{\widetilde{K}}$$

(This decomposition is multiplicity free, so $\mathcal{O}_1$ is a model orbit.)

Thank you!