Pro-p Iwahori Hecke algebra, inverse Satake transform and change of weight in characteristic p.
Singapour, 22 march 2016

Abstract

Let C be an algebraically closed field of positive characteristic p and let G be a reductive p-adic group. The classification of the admissible irreducible representations of G over C has been reduced to the classification of the supercuspidal ones by Abe, Henniart, Herzig and V. The proof uses the (injective) general Satake transform and the change of weight (proved using the pro-p Iwahori Hecke algebra of G). The image of the general Satake transform was an open question.

With N. Abe and Fl. Herzig, using the pro-p Iwahori Hecke algebra, we can now describe the image of the general Satake transform and give an explicit formula for its inverse. The explicit inverse Satake transform implies easily the change of weight.

Acknowledgments I am thankful to the organizers for inviting me a second time to give a talk in Singapour. Last time was during the Conference Modular Representation in April 2013. Noriyuki Abe and Florian Herzig who were also speakers at this conference, and we worked together at Singapour on the results that I am presenting now.

Preprint: A classification of irr adm mod p repr of p-adic red groups (AHHV) Accepted in the Journal of the American Mathematical Society

In preparation: Change of weight and inverse Satake formula (AHV)

Plan: 1. Classification 2. Satake homomomorphism 3. Weight, Hecke eigenvalue, supersingularity, change of weight, filtration.

p prime number, the motivation of this work is that in the future p-adic Langlands correspondence relating p-adic rep of the Galois group and p-adic automorphic representations of a reductive group, we will need some theory on the modulo p smooth rep of a red p-adic group. By modulo p, I mean that the coefficient field C is algebraically closed of characteristic p, by red p-adic group, I mean the group G of rational points of a linear connected reductive group $\underline{G}$ over a finite extension F of the field Q_p of p-adic numbers, or also of the field $F_p((X))$ of Laurent series over a finite field F_p of p elements in one variable X. Up to now the only case that is understood is the p-adic local Langlands for $GL(2, Q_p)$ where essentially all what I will say to-day is due to Barthel and Livne.

We fix a pair (S, B) where S is a maximal split torus, B a minimal parabolic subgroup (with the abuse that S, B are the F-rational points of such subgroups of $\underline{G}$). The groups S, B are unique modulo conjugation in G. Let Δ denote the set of simple roots of S with respect to B.

The **parabolic subgroups** P of G containing B are in bijection with the subsets $\Delta(P)$ of Δ. They have a Levi decomposition $P = MN$ with N the unipotent radical and a Levi component M containing S. Example: $B = ZU$ where Z is the centralizer of Z.

Then $(S, B \cap M)$ is a pair for M, and the set $\Delta(M)$ of simple roots of S with respect to $B \cap M$ is contained in Δ. The parabolic subgroup P is determined by $\Delta(M)$ that we denote also $\Delta(P)$. We get a bijection between the parabolic subgroups P of G containing B and the subsets of Δ.

A **representation** W of G over a C-vector space is smooth when the stabilizer of any vector of W in G is open. All the representations here will we smooth and over C.

Bad: G has an open pro-p group and there is no Haar measure on G with values in C.

Good: The representations are moderately ramified: if they are not 0, they have a non-zero vector invariant by any pro-p subgroup of G. A representation of a finite p-group over C has a fixed vector.

admissible: good: invariants by one open pro-p group is finite dimensional (irr implies adm ? known for C of char different from p)

The representations of G are obtained from representations W of smaller subgroups H by extension ${}^eW$ or induction c-$\mathrm{Ind}_H^G W$: . the smooth and compactly supported functions on the homogeneous space $H\backslash G$ with values in W satisfying the functional equation $f(hgk) = hf(g)$ for k in a small open sg of G, for the action of G by right translation (f smooth).

Two basic examples : $H = P = MN$, $P/N = M$. W a smooth representation of M.

Extension W extends to a representation of G trivial on N if and only if W is trivial on $Z \cap M'_\alpha$ for all $\alpha \in \Delta \setminus \Delta(M)$, where $M'_\alpha = \langle U_\alpha, U_{-\alpha}\rangle$.

$P_W = M_W N_W$ biggest parabolic such that inflation of W to P extends to P_W, extension is unique trivial on $N_W \subset N$. P_W corresponds to $\Delta(P) \cup \Delta(W)$ where $\Delta(W) = \{\alpha \in \Delta \setminus \Delta(P) \mid W \text{ is trivial on } Z \cap M'_\alpha\}$.

parabolic induction $\mathrm{Ind}_P^G W$ (W inflated to P), a left and a right adjoint, note that the quotient G/P is compact.

c-$\mathrm{Ind}_P^G 1$ or $St_P^G = \mathrm{Ind}_P^G 1 / \sum_{P \subsetneq Q \subset G} \mathrm{Ind}_Q^G 1$ called a Steinberg representation.

supercuspidal: irr adm not a subquotient of $\mathrm{Ind}_P^G W$ for some irr adm (after classification: kills left and right adjoint of Ind_P^G).

Thm 1 Classification St_P^G is irr. (GK in the split case, Ly in general)

For any triple P, W sc for M, $P \subset Q \subset P_W$, the representation

$$\mathrm{Ind}_{P_W}^G({}^eW \otimes \mathrm{St}_Q^{P_W})$$

of G is irr. adm. We denote it by $I(P, W, Q)$.

$I(P, W, Q) \simeq I(P', W', Q')$ is equivalent to $P = P', Q = Q', W \simeq W'$.

Any irr is isomorphic to a $I(P, W, Q)$.

$I(P, W, Q)$ is supercuspidal iff $P = G$, finite dimensional $I(B, W, G)$, $1 = I(P, 1, G)$, $St = I(B, 1, B)$.

What are the supercuspidal representations ? Existence ? For $GL(n, F)$ proof by global methods.

Ex $G = GL(2, F)$: supercuspidal, $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ with $\chi_1 \neq \chi_2$, $\chi \otimes \det$ and $(\chi \otimes \det) \otimes \mathrm{St}$ (Barthel-Livne).

G **finite** Z is a torus (G is quasi-split), if V is an irreducible representation of G then V_U has dimension 1. Let ψ_V denote the corresponding character of Z is the lowest weight vector, $V^{U_{op}} \simeq V_U$ (via the quotient map $v \to \overline{v} : V \to V_U$) as representations of $Z \cap K$. The G-stabilizer of the kernel V_U is a parabolic subgroup Q contained in the parabolic subgroup $P(\psi_V)$ associated to $\Delta(\psi_V) = \{\alpha \in \Delta \mid \psi_V \text{ is trivial on } Z \cap \langle U_\alpha, U_{-\alpha}\rangle\}$. If Q corresponds to $\Delta(V)$, then $(\psi_V, \Delta(V))$ determines the isomorphism class of V. Any such pair (ψ_V, Δ_V) corresponds to an irreducible representation V. There are no irreducible supercuspidal representations if $G \neq Z$. For all this Carter-Lusztig, Cabanes. The set $\Delta(V)$ measures the "irregularity" of the representation: The Steinberg representation $\Delta(St) = \emptyset$, the trivial representation $\Delta(1) = \Delta$.

This is invisible in the statement, but the key of the proof of the classification is

the **compact induction**, the **Satake homomorphism**, and the **pro-p Iwahori Hecke algebra**.

We fix a **special parahoric subgroup** K of G (not unique modulo conjugation). Although it is for $GL(n, F)$ where K is simply a conjugate of the maximal open compact subgroup $GL(n, O_F)$, O_F the ring of integers of F. We put $X^0 = X \cap K$ for a sg of G. The group M^0 is special parahoric in M. The group Z^0 is the unique parahoric subgroup of G. It is normal in Z with a commutative quotient Z/Z^0 even when G is not quasisplit (Z is not a torus) (Haines-Rostami), has a unique pro-p Sylow subgroup $Z(1)$.

Satake homomorphism :

$$f \mapsto S(f)(z) = \sum_{x \in U^0 \backslash U} f(xz) : H(G, K) \to H(Z, Z^0)$$

It is injective. As $H(Z,Z^0)=C[Z/Z^0]$ is commutative, $H(G,K)$ is commutative.

When C of characteristic 0, normalized by the module of B, its image is $C[Z/Z^0]^W$ where W is the Weyl group (quotient of the G-normalizer of Z by Z), at the origin of the definition of the L-group by Langlands. When G is simple split and simply connected, Lusztig-Kato formula gives a formula for the inverse of S on $C[Z/Z^0]^W$.

Without normalization, the image is $C[Z^+/Z^0]$ where Z^+ is the set of dominant elements, the natural map $Z^+/Z^0 \to K\backslash G/K$ is an isomorphism. (Hen-V). $C[Z/Z^0]$ is a localization of $C[Z^+/Z^0]$. Valid over $\mathbb{Z}$. When G is simple split and simply connected, there is also a formula for the inverse of S on $C[Z^+/Z^0]$ (Her).

We need something to study more generally the representations.

$$\text{c-Ind}_K^G V$$

for an irreducible representation V of K. Understand their structure is not easy but of fundamental importance. (The first person who realized this for complex representations is in this room !). We can extend the Satake homomorphism to study the space $H(G,K,V,V')$ of G-intertwiners $\text{c-Ind}_K^G V \to \text{c-Ind}_K^G V'$ for two irreducible representations V,V' of K, we can break them and understand some pieces.

G-Intertwiners $\mathrm{Hom}_G(\text{c-Ind}_K^G V, \text{c-Ind}_K^G V')$ isomorphic to the spherical Hecke space $H(G,K,V,V')$ of compactly supported functions $f: G \to \mathrm{Hom}_C(V,V')$ satisfying the functional equation $f(k'gk)=k'f(g)k$. When $V=V'$ $\mathrm{End}_G\, \text{c-Ind}_K^G V$ isomorphic to the spherical Hecke algebra $H(G,K,V)$ (convolution algebra).

$Z^+/Z^0 \simeq K\backslash G/K$ and a double class KzK with $z\in Z^+$ supports an intertwiner iff z belongs to

$$Z(V,V')=\{z\in Z \mid z.\psi_V=\psi_{V'},\ \langle\alpha, v(z)\rangle>0 \ \text{ for } \ \alpha\in\Delta(V)\setminus\Delta(V')\cup\Delta(V')\setminus\Delta(V)\},$$

$v: Z\to X_*(S)\otimes\mathbb{Q}$. Modulo a scalar, there is a unique intertwiner T_z of support KzK for $z\in Z^+(V,V')=Z^+\cap Z(V,V')$. Basis of $H(G,K,V,V')$ is $(T_z)_{z\in Z^+(V,V')/Z^0}$.

If $V=V'$, $Z(V,V)=Z_{\psi_V}$ is the Z-normalizer of the character ψ_V of Z^0 and depends only on ψ_V.

Thm 2 Satake homomorphism The map

$$f\mapsto S(f)(z)=\sum_{x\in U^0\backslash U}\overline{f}(xz): H(G,K,V,V')\to H(Z,Z^0,V_{U^0},V'_{U^0})$$

is injective. It respects the product if $V=V'$. Its image has the basis $(\tau'_z)_{z\in Z^+(V,V')/Z^0}$ where

$$\tau'_z=\tau_z\prod_{\alpha\in\Delta'(V')\setminus\Delta'(V)}(1-\tau_{a_\alpha}).$$

Here a_α is the generator of $(Z\cap M'_\alpha)/(Z^0\cap M'_\alpha)$ such that $v(\alpha)\in\mathbb{Q}_{<0}\alpha^\vee$ and $\Delta'(V)=\{\alpha\in\Delta(V)\mid \psi_V \ \text{ trivial on } \ Z^0\cap M'_\alpha\}$.

$$S^{-1}(\tau_z)=\sum_{x\in Z^+\cap z\prod_{\alpha\in\Delta'(V)\cap\Delta'(V')}a_\alpha^{\mathbb{N}}}T_x$$

after a good normalization of τ_z, T_x

If $V=V'$, the algebra $H(Z,Z^0,V_{U^0})=H(Z_{\psi_V},Z^0,\psi_V)$ is the localisation of $H(G,K,V)\simeq H(Z^+_{\psi_V},Z^0,\psi_V)$. The algebra depends only on ψ_V, is commutative when G is quasi-split, but not in general. But it is a finitely generated module over the center $Z(G,K,V)$, and the center is a finitely generated algebra. When G is semisimple simply connected split, Herzig using Lusztig-Kato determined S^{-1}.

If $\Delta(V')\subset\Delta(V)$, $H(G,K,V,V')\simeq H(Z^+(V,V'),Z^0,V_{U^0},V'_{U^0})$. But τ'_z is not contained in Z^+ if $\Delta'(V')\setminus\Delta'(V)\neq\emptyset$.

Ex $GL(2,F)$, $V = 1, V' = St$ (Kisin Lemma 1.5.5, 2009 proof of Fontaine Mazur conjecture for $GL(2)$)

The proof of (new) uses the pro-p Iwahori algebra $H(G,I)$ and the representation

$$\mathfrak{X} = \mathrm{Ind}_B^G \, \text{c-}\mathrm{Ind}_{Z(1)}^Z 1$$

which contains an element $f_V \in \mathfrak{X}^I$ such that $\text{c-}\mathrm{Ind}_K^G V \simeq CGf_V$ and $f_V H(G,I) \simeq (\text{c-}\mathrm{Ind}_K^G V)^I$.

The parabolic induction relates these representations and those of the Levi subgroups

The Satake homomorphism S and the homomorphism $\text{c-}\mathrm{Ind}_K^G \hookrightarrow \mathfrak{X}$ factorize

$$H(G,K,V,V') \to H(M, M^0, V_{N^0}, V'_{N^0}), \quad \text{c-}\mathrm{Ind}_K^G V \hookrightarrow \mathrm{Ind}_P^G(\text{c-}\mathrm{Ind}_{M^0}^M V_{N^0}).$$

V_{N^0} irr repr of M^0 corresponding to $(\psi_V, \Delta(M) \cap \Delta(V))$, replace U by N and $z \in Z$ by $m \in M$ in the formula for the Satake homomorphism.

Let W be a non zero rep of G. An irreducible representation V of K such that $\mathrm{Hom}_K(V,W) \neq 0$ is called a **weight** of W. There exists always a weight because V is smooth. By Frobenius reciprocity $\mathrm{Hom}_K(V,W) = \mathrm{Hom}(\text{c-}\mathrm{Ind}_K^G V, W)$. Every irreducible representation W of G is isomorphic to a subquotient of $\mathfrak{X}$, a representation parabolically induced from the Borel subgroup B !

$\mathrm{Hom}(\text{c-}\mathrm{Ind}_K^G V, W)$ is a right module over the algebra of G-intertwiners of $\text{c-}\mathrm{Ind}_K^G V$. When $Z(G,K,V)$ admits an eigenvector in $\mathrm{Hom}(\text{c-}\mathrm{Ind}_K^G V, W)$, the eigenvalue is called a **Hecke eigenvalue** of $Z(G,K,V)$ (or of V) in W. When W is irreducible, V is a weight of W with Hecke eigenvalue χ means that W is a quotient of

$$\chi \otimes_{Z(G,K,V)} \text{c-}\mathrm{Ind}_K^G V.$$

Any weight of W admits an eigenvalue in W when W is admissible.

We can now introduce the notions of supersingularity and explain the strategy of the proof of the classification theorem.

A character $\chi : Z(G,K,V) \to C$ is **supersingular** if it does not extend to $Z(M, K \cap M, V_{N \cap K})$ for $M \neq G$. W irr. adm. is called **supersingular** when all its Hecke eigenvalues are supersingular (for all weights V).

$\Delta(\chi) = \{\alpha \in \Delta \mid \chi(z_\alpha) = 0\}$ for any $z_\alpha \in S, v((z_\alpha) > 0, \langle \beta, v((z_\alpha) \rangle = 0 \; \beta \in \Delta - \backslash\{\alpha\}\}$ corresponds to the smallest M such that χ extends to a (supersingular) character of $Z(M, N^0, V_{N^0}, V'_{N^0})$.

We prove the theorem with W supersingular, not supercuspidal ! After the theorem (with supersingular) we deduce that supersingular=supercuspidal.

The irreducibility of $\mathrm{Ind}_{P_e}^G({}^eW \otimes \mathrm{St}_Q^{P_e})$ uses that $\mathrm{Ind}_P^G W$ is irr iff W contains a weight with $\Delta(V) \subset \Delta(P)$. This is the motivation for the change of weight. We try to lower $\Delta(V)$. Recall that $Z(G,K,V)$ depends only on ψ.

Change of weight When $\psi_{V'} = \psi_V$ and $\Delta(V') = \Delta(V) - \{\alpha\}$

$$\chi \otimes \text{c-}\mathrm{Ind}_K^G V \simeq \chi \otimes \text{c-}\mathrm{Ind}_K^G V'$$

except if α orthogonal to $\Delta(\chi)$, ψ is trivial on $Z^0 \cap M'_\alpha$ and $\chi(\tau_{a_\alpha}) = 1$.

The proof uses the description of the image of the Satake homomorphism for the two pairs (V,V') and (V',V).

$\mathrm{Ind}_{P(W)}^G({}^eW \otimes \mathrm{St}_Q^{P(W)})$ determines P, P_W, W, Q. The proof uses that the weight and eigenvalues of $\mathrm{Ind}_P^G W'$ are determined by those W' that $\mathrm{Ind}_P^G W'$ determines W' (for any rep W').

Finally the exhaustion... we are all exhausted !

Generalization of: $\mathrm{Ind}_B^G 1$ filtration quotients St_Q^G, trivial as a subrepresentation, St as a quotient, but $\chi \otimes \text{c-Ind}_K^G 1$ as a filtration with the same quotients in the inverse order St as a sub and 1 as a quotient.

$\chi \otimes \text{c-Ind}_K^G V$ has a natural **filtration** with quotients $\mathrm{Ind}_{P_e}^G {}^e(\chi \otimes \text{c-Ind}_{M^0}^M V_{N^0}) \otimes \mathrm{St}_Q^{P_e}$ for $P \subset Q \subset P_e$ in the inverse order (with P as a sub and P_e as a quotient).

P is the smallest parabolic subgroup such that χ extends to a supersingular character of $Z(M, M^0, V_{N^0})$,

P_e is the biggest parabolic subgroup such that $\chi \otimes \text{c-Ind}_{M^0}^M V_{N^0}$ extends to a representation ${}^e(\chi \otimes \text{c-Ind}_{M^0}^M V_{N^0})$ of P_e trivial on N.