Towards homotopy methods in representation theory

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Langlands correspondences: Classical

Basic definitions:

- G a reductive group over k, e.g., $GL(n), Sp(2n)$
- $G^\vee$ the complex dual group, e.g., $GL(n,\mathbb{C}), SO(2n+1,\mathbb{C})$
- $W_k \subset \mathrm{Gal}(\bar{k}/k)$ the absolute Weil group of k
- ${}^LG = \hat{G} \rtimes W_k$ the L-group
- $L_k \twoheadrightarrow W_k$ 'the' Langlands group

The arithmetic Langlands correspondence is, roughly,

- k a local field of characteristic 0, there is a finite to 1 map:

$$\mathrm{Irr}(G(k)) \longrightarrow \{\psi : L_k \times SL_2(\mathbb{C}) \to {}^LG\}/G^\vee\text{-conj},$$

  giving a partition $\Pi_{\mathrm{unit}}(G(k)) = \sqcup_\psi \Pi_\psi$ with maps $\Pi_\psi \to \mathrm{Irr}(S_\psi)$.
- k a global field of characteristic 0, we have

$$L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}_k)) = \bigoplus_{\psi\ \mathrm{disc}} \bigoplus_{\pi\in\Pi_\psi} \pi^{m(\pi),\psi}$$

  where $\Pi_\psi = \otimes' \Pi_{\psi_v}$, the restricted direct product over all places v.

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Langlands correspondences: Covering groups

The theory of automorphic forms has involved the representation theory of *covers* of reductive groups, which are often not algebraic. Examples:

- Weil representation and theta correspondence on metaplectic groups;
- Shimura lifts on covers of general/special linear groups
- Weissman's *L*-group for Brylinski-Deligne extensions

Theorem (Gan-Savin, 2012)

Let ψ be a nontrivial additive character of k, μ the roots of unity in k with an embedding $\epsilon : \mu \hookrightarrow \mathbb{C}^\times$. Then there is a bijection

$$Irr_\epsilon(Mp_{2n}) \leftrightarrow Irr(SO_{2n+1}(k)) \sqcup Irr(SO_{2n+1}(k))$$

where the LHS denotes representations π such that $\pi|_\mu = \epsilon$.

Question: How do covering groups fit into the Langlands correspondence?

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Langlands correspondences: n-dimensional

A nonarchimedean local field is a complete discrete valued field with finite residue field, e.g., $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Define an *n-dimensional local field* inductively to be one whose residue field is an $(n-1)$-dimensional local field, e.g., $\mathbb{Q}_p((t)), \mathbb{F}_p((t_1))((t_2))$.

Theorem (Kapranov, 1992)

The Langlands correspondence is a stack on the Waldhausen space—a bisimplicial category—associated to the category of (pure) motives.

dim n	n-categories	objects
0	k	subsets of k
1	Vect$/k$	V/k
2	2-Vect$/k$	Vect-modules$/k$
$\vdots$	$\vdots$	$\vdots$

For the $n = 2$, Parshin's version (2013) of Kapranov's proposed correspondence reads:

$\{d\text{-dim reps of } \mathrm{Gal}(\bar{k}/k)\} \leftrightarrow \{\text{Irred. 2-reps of } GL(2d, k)\}$

Problem: n-categories are not well-developed. (But ∞-categories are!)

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This talk: Motives most naturally live in an $(\infty, 1)$-category. How should the automorphic side reflect this structure?

Stacks for geometers

Why stacks? They (1) solve moduli problems, (2) keep track of nontrivial automorphisms in quotient groups.

$$F \qquad\qquad \text{Gpd}$$
$$\downarrow_p \qquad\qquad \uparrow_{p^{-1}}$$
$$C \qquad\qquad C$$

Definition: A *stack* (in groupoids) over a category C is a category k fibred in groupoids such that

- isomorphisms are a sheaf and
- descent datum is effective

In other words, p^{-1} is a sheaf of groupoids on C. (Really a 2-sheaf.)
We call a stack *algebraic* (or Artin) if

- the diagonal $F \to F \times F$ is representable, quasi-compact, and separated,
- There is a smooth surjective morphism from a scheme $X \to F$

Example: $C = \text{Spec } \mathbb{Z}$, $F = \mathcal{M}_g$ smooth curves of fixed genus $g \geq 2$ is an algebraic stack.

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Stacks for representation theorists

Example: Let X be an S-scheme with an action of an algebraic group $\mathbf{G}$, with k points $G =: \mathbf{G}(k)$. The *quotient stack* is the contravariant functor

$$[X/\mathbf{G}] : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Gpd}$$

associating to an S-scheme Y the category of principle G-bundles over Y with a G-equivariant morphism to X.

Example If $X = \mathrm{Spec}(k) = *$, then $[*/\mathbf{G}]$ is the moduli stack of principle G-bundles over S, called the *classifying stack* $B\mathbf{G}$. It is known that

$$\mathrm{Rep}(G) \simeq \mathrm{QC}(BG),$$

where Rep denotes the category of smooth, finite-dimensional complex representations and QC the category of quasicoherent sheaves.

Theorem (Bernstein, 2014)

Let $\mathbf{G}_i$ be the pure inner forms of $\mathbf{G}$ over a nonarchimedean local field k. Then

$$\mathrm{Irr}(\mathrm{QC}(B\mathbf{G}(k)) = \coprod \mathrm{Irr}(G_i)$$

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Stacks for topologists

Definition: Let Δ be the category whose objects are the relations $0 \to 1 \to \cdots \to n$ for $n \geq 0$, and the morphisms are order-preserving set functions. Then a *simplicial set* is a contravariant functor

$$\Delta^{\mathrm{op}} \to \mathrm{Sets}$$

and a *simplicial presheaf* over C is a contravariant functor

$$C^{\mathrm{op}} \to \mathrm{sSets}.$$

i.e., a simplicial object in Pre/C.
Definition: A simplicial presheaf k is a *stack* if for any hypercovering H of any $X \in C$ the natural morphism

$$F(X) \to \mathrm{holim}_{\Delta} F(H_n)$$

is an equivalence of simplicial sets. Inductively, a stack is 0-algebraic if k is a scheme, and n-algebraic if there is a scheme $X \to F$ with a smooth $(n-1)$ algebraic epimorphism to k (we'll not define this here). Finally, an *algebraic stack* is a stack that is n-algebraic for some n.

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Definition: A simplicial presheaf k is a *stack* if for any hypercovering H of any $X \in C$ the natural morphism

$$F(X) \to \text{holim}_{\Delta} F(H_n)$$

is an equivalence of simplicial sets. Inductively, a stack is 0-algebraic if k is a scheme, and n-algebraic if there is a scheme $X \to F$ with a smooth $(n-1)$ algebraic epimorphism to k (we'll not define this here). Finally, an *algebraic stack* is a stack that is n-algebraic for some n.

Stacks for us: What

A closed model structure on a category is a specified class of maps (fibrations, cofibrations, weak equivalences) satisfying certain axioms. By Lurie, one may assign an $(\infty, 1)$-category to a given model category.

Our construction is now straightforward: Consider simplicial sheaves of sets on $B\mathbf{G}$, then:

Proposition (W.)

The category $\mathrm{sShv}(\mathrm{B}\mathbf{G}(\mathrm{k}))$ *has a closed model structure with the model structure of Joyal, in which*

- *Cofibrations are the monomorphisms,*
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Thus there exists an $(\infty, 1)$*-category underlying this model category.*

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Stacks for us: Why

The Langlands correspondence can be formulated using motives. (Indeed, L_k was inspired by Grothendieck's pure motives.) Morel and Voevodsky developed a homotopy theory of schemes, whose construction goes like this:
Starting with the category of smooth schemes over a field k, form

$$\mathrm{Sm}/k \to \mathrm{sShv}(\mathrm{Sm}/k) \to \mathrm{sShv}(\mathrm{Sm}/k)_{\mathbb{A}^1}$$

the final term being localization with respect to projections $X \times \mathbb{A}^1 \to X$, we call this the (unstable) motivic homotopy category of schemes.

Theorem (Dugger, 2000)

There is a Quillen equivalence of model categories

$$\mathrm{sShv}(\mathrm{Sm}/k)_{\mathbb{A}^1} \xrightarrow{\sim} \mathrm{sPre}(\mathrm{Sm}/k)_{\mathbb{A}^1}$$

where $\mathrm{sPre}(\mathrm{Sm}/k)$ *is the universal model category associated to Sm/k.*

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We mention in passing where stacks have arisen in relation to the Langlands correspondence:

- Nash stacks as a setting for the relative trace formula, global version of Bernstein (Sakellaridis)
- Moduli stack of Higgs bundles in proof of the fundamental lemma (Ngô)
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Stacks for us: Questions

- What should reflect $\mathbb{A}^1$-localization of the representation theory side?
- Can this construction work for covering groups, i.e., does the following hold:

$$\mathrm{Irr}(\mathrm{QC}(B\tilde{\mathbf{G}}(k)) \overset{?}{=} \coprod \mathrm{Irr}(\tilde{G}_i)$$

  Do we get $\mathrm{Irr}(\mathrm{QC}(BMp_2n(k)) = \mathrm{Irr}_\epsilon(Mp_{2n})$? (Probably not exactly.)
- Bernstein's construction covers pure inner forms of $\mathbf{G}$. What about Kaletha's rigidified/extended pure inner forms, for when $\mathbf{G}$ is not quasiplit over k?
- How does this relate to Schneider's equivalence of derived categories over p-adic fields:

$$D(\mathrm{Rep}(G)) \xrightarrow{\sim} D(H(G, I(1)) - \mathrm{Mod})$$

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