Uniform Error Estimates for Finite Differences Methods Applied to Fluid Motion with Interfaces

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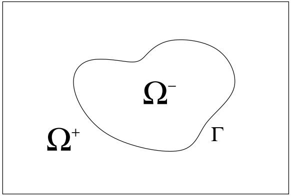
Finite differences with regular grids, irregular boundaries
with jump conditions at the bdry or interface
especially fluid flow with viscosity, Navier-Stokes eq'ns

Estimates in maximum norm for several equations

(1) Discrete Poisson equation, gain in regularity
(2) Application to the immersed interface method
(3) Discrete diffusion equation
(4) Error estimates for Navier-Stokes with a moving interface, velocity and pressure, neglecting the error in interface position validating the general approach in a simple setting

The Problem and the Result

Navier-Stokes eq'ns in $\mathbb{R}^2$ or $\mathbb{R}^3$
periodic boundary conditions
interface $\Gamma(t)$, a closed surface
force $\mathbf{f}$ on Γ acts on fluid
Re $= O(1)$



$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f} \delta_\Gamma$$
$$\nabla \cdot \mathbf{u} = 0$$

The interfacial force $\mathbf{f}$ amounts to jumps in $\nabla \mathbf{u}$ and p.

Result. We assume $\Gamma(t)$ known; f, $\nabla_{\tan} f$ known to $O(h^2)$.
With one scheme based on the immersed interface method,
with truncation error $O(h)$ at Γ and $O(h^2)$ elsewhere,
the error in velocity and pressure is uniformly about $O(h^2)$.

Estimates in maximum norm for finite difference versions of equations such as

$$\Delta u = f$$

$$u_t = \Delta u + f$$

on a rectangular grid in $\mathbb{R}^d$, any dimension d
Estimates in L^2 are the most natural.
For problems with limited smoothness, e.g. with interfaces,
largest error is typically on a small set
L^∞ is a more meaningful measure of error than L^2
methods are designed to control maximum truncation error

We get estimates of the form $\Delta_h u^h = f^h \to \|u^h\| \leq C|f^h|$
Equations are linear, so that estimates apply to errors:
If $\Delta_h u^h = f^h$ and $\Delta_h u^{exact} = f^h + \varepsilon^h$, $\varepsilon^h =$ truncation error,
we subtract to get $\|u^h - u^{exact}\| \leq C|\varepsilon^h|$

Discrete Laplacian ($d = 2$) with spacing h:

$$\Delta_h u(ih, jh) = \left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\right)/h^2$$

$$\Delta_h u = D_1^+ D_1^- u + D_2^+ D_2^- u$$

$D_1^+ u = \left(u_{i+1,j} - u_{i,j}\right)/h\,, \quad D_1^- u = \left(u_{i,j} - u_{i-1,j}\right)/h$
For smooth u, $\Delta_h u = \Delta u + O(h^2)$ from Taylor expansion
Does inverting Δ_h gain differences?

For the exact problem $\Delta u = f$, periodic, average zero
we have sharp L^p estimates, $1 < p < \infty$

$$\|D^k u\|_{L^p} \leq C_p \|f\|_{L^p}\,, \qquad k \leq 2$$

There are also "sharp" estimates in Hölder norms $C^{k+\lambda}$
(Schauder estimates) but **not** L^∞

Estimate for the Discrete Laplacian. Suppose u_h and f_h are periodic grid functions on $\mathbb{R}^d$ with average zero and $\Delta_h u_h = f_h$. Then in maximum norms

$$\|u_h\| + \|D_h u_h\| \leq C\|f_h\|\,,$$

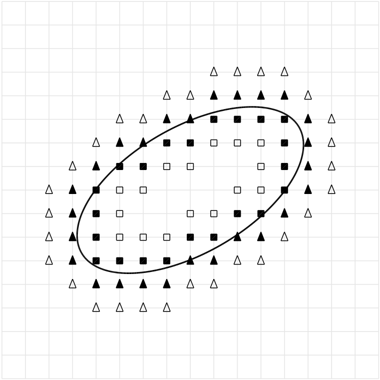
$$\|D_h^2 u_h\| \leq C|\log h|\|f_h\|$$

with C independent of h, $D_h =$ any first difference.
The log factor cannot be improved.
E.g., $u(x,y) = (x^2 - y^2)\log r$

This can be proved using estimates for a discrete Green's function.
J.T.B. and A. Layton, CAMCoS (2006)
The discrete maximum principle estimates u_h, but not differences.
V. Thomée proved discrete Schauder estimates (1968).
Estimates for finite element spaces are better known.

Poisson problem with an interface

$$\Delta u_- = f_- \quad \text{in } \Omega_- , \qquad \Delta u_+ = f_+ \quad \text{in } \Omega_+ ,$$
$$[u] = g_0 \quad \text{on } \Gamma , \qquad [\partial_n u] = g_1 \quad \text{on } \Gamma$$



Poisson problem with an interface

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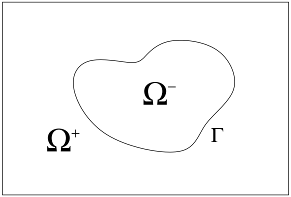


$\Delta_h u(ih) - \Delta u(ih) = O(h^2)$
at regular points,
away from Γ

$\Delta_h u(ih) - \Delta u(ih)$ is large
at irregular points,
where Δ_h crosses Γ

If we improve the truncation error to $O(h)$ at the irreg. pts.
using the Immersed Interface Method, then
the error in u is uniformly $O(h^2)$, and
the error in $D_h u$ is uniformly $O(h^2 |\log h|)$

Immersed Interface Method, R. LeVeque, Z. Li; A. Mayo

Poisson problem with interface:

$$\begin{aligned} \Delta u_- &= f_- \quad \text{in } \Omega_- , & \Delta u_+ &= f_+ \quad \text{in } \Omega_+ , \\ [u] &= g_0 \quad \text{on } \Gamma , & [\partial_n u] &= g_1 \quad \text{on } \Gamma \end{aligned}$$

Find C_h so that $\Delta_h u^{exact} - f^{exact} = C_h + O(h)$. Solve

$$\Delta_h u_h = f_h + C_h$$

$C_h = 0$ at regular grid points; truncation error $O(h^2)$
$C_h \neq 0$ at irregular grid points near Γ, e.g.
If $x_j \in \Omega_-$, $x_{j+1} \in \Omega_+$, $h_+ = x_{j+1} - x^*$

$$v_{xx}(x_j) = \frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} - \frac{1}{h^2}\left([v] + h_+[Dv] + \frac{h_+^2}{2}[D^2 v] \right) + O(h)$$

Zhilin Li & K. Ito, The Immersed Interface Method..., SIAM, 2006

With corrections, $\Delta_h(u_h - u) = O(h)$ at the irregular points since the C_h's cancel. It is $O(h^2)$ at the regular points.

Theorem. With truncation error $O(h^2)$ at regular points and $O(h)$ truncation error at irregular points,

$$|u_h(jh) - u(jh)| \leq C_0 h^2 ,$$
$$|D_h u_h(jh) - D_h u(jh)| \leq C_1 h^2 |\log h| .$$

Brief summary of proof (J.T.B. and A. Layton, CAMCoS '06):
The $O(h)$ truncation error at irregular points has the form

$$F_0 + D_h^{(1)} F_1 + \cdots + D_h^{(d)} F_d, \qquad F_i = O(h^2)$$

Then $\quad \Delta_h(u_h - u) = F_{reg} + F_0 + \nabla_h \cdot (F_1, \ldots, F_d)$

The estimate for $u_h - u$, has a gain of one deriv
for $D_h(u_h - u)$, gain of two derivs, log factor

Lemma. Suppose f^{irr} is a grid function on Ω such that $f^{irr} \neq 0$ only at irregular points (near the interface). Then there are periodic grid functions F_k so that $f^{irr} = F_0 + \sum_{k=1}^{d} D_k^- F_k$ and

$$\|F_k\|_{max} \leq Ch\|f^{irr}\|_{max} .$$

Key example in one dimension:
$F(ih) = 0$ for $i \leq 0$; $F(ih) = 1$ for $i > 0$
Set $D_h^+ F(ih) = [F((i+1)h) - F(ih)]/h$
Then $D_h^+ F(ih) = 1/h$ for $i = 0$; $D_h^+ F(ih) = 0$ for $i \neq 0$
$D_h^+ F \to F$ gains a factor of h in maximum norm.

To prove in dimension d, write f^{irr} locally
as the difference of its sum in a direction
transverse to Γ; use partition of unity

Similar lemmas in L^2: Hackbusch '81, Stevenson '91

Gain in Regularity for Parabolic Difference Equations

Suppose we approximate (using backward Euler)

$$u_t \;=\; \Delta u \;+\; f\;, \quad u(\cdot,0) = 0$$

by $\qquad u^{n+1} - u^n = \tau \Delta_h u^{n+1} + \tau f^{n+1}$

or $\qquad u^{n+1} = (I - \tau\Delta_h)^{-1}\left(u^n + \tau f^{n+1}\right)$

with $\qquad t_n = n\tau,\;\; \tau = \text{ time step } = ch$

Then

$$\|u^n\| + \|D_h u^n\| \le C_1 \sup_{t\le T} \|f(\cdot,t)\|$$

$$\|D_h^2 u^n\| \le C_2\,(1 + |\log h|) \sup_{t\le T} \|f(\cdot,t)\|$$

Similar statements hold for a class of time-stepping methods...

Interpretation: $u^{\text{computed}} - u^{\text{exact}}$ and differences

are bounded by maximum truncation error.

J.T.B. "Smoothing properties...", SINUM 2009.

Gain in Regularity for Diffusive Difference Equations

Discretize $u_t = \Delta u$ in $\mathbb{R}^d \times [0, T]$
grid spacing h, time step τ
use Δ_h, usual second-order Laplacian, $\Delta_h = \sum_{\nu=1}^{d} D_\nu^+ D_\nu^-$
use implicit time stepping, $\tau = O(h)$
for a single step method, $u^{n+1} = Su^n$ and $u^n = S^n u^0$
Crank-Nicolson method (second-order accurate):

$$\begin{aligned} u^{n+1} - u^n &= (\tau/2)(\Delta_h u^{n+1} + \Delta_h u^n)\,, \\ S = s(\tau\Delta_h) &= (1 + \tfrac{\tau}{2}\Delta_h)(1 - \tfrac{\tau}{2}\Delta_h)^{-1} \end{aligned}$$

The best results are for L-stable methods, i.e.

$|s(z)| \leq 1$ for Re $z \leq 0$ (A-stable)
$s(z) \to 0$ as $z \to \infty$

Examples: backward Euler, TGA (an improvement of CN);
one SDIRK2 (Runge-Kutta); BDF2 (but multi-step)
CN is A-stable, but not L-stable; $s(\infty) \neq 0$.

Main Result. For an L-stable, single step method with operator norm on $L^\infty(\mathbb{R}^d_h)$,

$$\|S^n\| \le C_0$$
$$\|D_h S^n\| \le C_1 (n\tau)^{-1/2}$$
$$\|D_h^2 S^n\| \le C_2 (n\tau)^{-1}$$

for $n\tau \le T$, with constants indep't of h, τ; similarly BDF2.
For CN, more limited estimates: $\|S^n\| \le C_0$ and

$$\|D_h R S^n\| \le C_1 (n\tau)^{-1/2}, \quad R = (1 - \tfrac{\tau}{2}\Delta_h)^{-1}$$

J.T.B., "Smoothing Properties...", SINUM, 2009

Related work: Aronson ('63), Widlund ('66) for $\tau = O(h^2)$;
Ashyralyev & Sobolevskii; Thomée et al. for finite elements
Michael Pruitt, Numer. Math. ('14) generalization to
parabolic operators with variable coefficients

Proof of the Main Result

With $S =$ time step operator, L-stable, single step, in max norm

$$\|S^n\| \le C_0$$
$$\|D_h S^n\| \le C_1 (n\tau)^{-1/2}$$
$$\|D_h^2 S^n\| \le C_2 (n\tau)^{-1}$$

Use the point of view of analytic semigroups, as in

Thomée, Galerkin FEM for Parabolic Problems

Ashyralyev & Sobolevskii, ...Parabolic Difference Eq'ns

Proof in three steps:

(1) For the semidiscrete equation $u_t = \Delta_h u\,, \quad u = u(jh, t)$
estimate u and its differences for **complex** t,

$$\|D_h^m e^{\Delta_h t}\| \le C_m |t|^{-m/2} \text{ on } L^\infty(\mathbb{R}_h^d)$$

(2) Estimate $D_h^m (z - \Delta_h)^{-1}$ using (1).

(3) Write $s^n(z)$ as a contour integral using $(z - \Delta_h)^{-1}$;
estimate $D_h^m s^n(z)$ on $L^\infty(\mathbb{R}_h^d)$ for n large.

Step 1. Semidiscrete: $u_t = \Delta_h u\,, \quad u(t) = e^{\Delta_h t} u^0$

Show $\|D_h^m e^{\Delta_h t}\| \le C_m |t|^{-m/2}$ on $L^\infty(\mathbb{R}_h^d)$, for complex t

t in a sector, $\{t = t_1 + it_2 : t_1 > 0, |t_2| \le M t_1\}$

Use $\quad g_t = \Delta_h g\,, \quad g(jh, 0) = \delta_{j0}$

$$u(jh, t) = \textstyle\sum_\ell g(jh - \ell h, t) u^0(\ell h)$$

Estimate $D_h^m g$ in discrete L^1 using transform (F. John, '52)

$$f(jh) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat f(\xi) e^{ij\xi}\, d\xi\,, \qquad \hat f(\xi) = \sum_{j\in\mathbb{Z}} f(jh) e^{-ij\xi}$$

in one dimension; $\hat g(\xi, t) = \exp(-4t\sin^2(\xi/2)/h^2)$; e.g.,

$$|g(jh,t)| \le \frac{C}{j^2}\int_{-\pi}^{\pi} |\hat g''(\xi,t)|\, d\xi \le \frac{C'}{j^2}\frac{|t|^{1/2}}{h}$$

$$\left(D_h^+ g\right)\hat{}(\xi) = (2i/h)\, e^{i\xi/2}\sin(\xi/2)\hat g(\xi)$$

Step 2. Estimate the resolvent of Δ_h:

$$(z-\Delta_h)^{-1} = \int_0^\infty e^{-zt} e^{\Delta_h t}\, dt$$

Moving the ray in the t-sector, we get

$$\begin{aligned} \|(z-\Delta_h)^{-1}\| &\leq C_0 |z|^{-1}, \\ \|D_h(z-\Delta_h)^{-1}\| &\leq C_1 |z|^{-1/2}, \\ \|D_h^2(z-\Delta_h)^{-1}\| &\leq C_2(1+|\log|z||+|\log h|) \end{aligned}$$

for z outside a sector about $z<0$

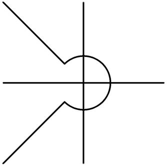
For periodic functions, mean value zero,

$$\|(\Delta_h)^{-1}\| \leq C_0\,,\ \|D_h(\Delta_h)^{-1}\| \leq C_1\,,\ \|D_h^2(\Delta_h)^{-1}\| \leq C_2(1+|\log h|)$$

Step 3. Estimate $S^n = s(\tau\Delta_h)^n$ and differences:

$$D_h^\alpha s(\tau\Delta_h)^n = \frac{1}{2\pi i}\int_\Gamma s(z)^n D_h^\alpha (z - \tau\Delta_h)^{-1} dz$$

use contour Γ, radius $O(1/n)$
use resolvent estimates
from Step 2
use assumptions on s:



$s(z)$ analytic in a sector about $z < 0$,
$s(z) = 1 + z + O(z^2) \quad$ as $z \to 0$,
$|s(z)| \leq (1 + c_1|z|)^{-1}$, z in sector about $z < 0$
(from consistency and L-stability)

J.T.B., "Smoothing Properties...", SINUM, 2009

Discretizing the Navier-Stokes equations

For exact solutions with periodic boundary conditions

$$u_t + u \cdot \nabla u + \nabla p = \Delta u + F\,, \quad \nabla \cdot u = 0$$

Use P , the L^2-projection on divergence-free vector fields

$$u_t + P(u \cdot \nabla u) = \Delta u + PF$$

where $\qquad Pw = w - \nabla \Delta^{-1} \nabla \cdot w$

To discretize in space, use centered diff's for ∇, e.g.

$$\frac{\partial u}{\partial x_1} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

We use the usual discrete Laplacian and get an
"approximate projection".
(A staggered grid (MAC) would avoid this.)

Use centered differences for grad and div, $\nabla \approx \nabla_h$
Suppose we define $\Delta_w u = \nabla_h \cdot \nabla_h u$,
then Δ_w is the "the wide Laplacian", in 2D

$$\Delta_w u = \frac{u_{i+2,j} + u_{i-2,j} + u_{i,j+2} + u_{i,j-2}}{4h^2}$$

$P_0 = I - \nabla_h \Delta_w^{-1} \nabla_h \cdot$ is a projection,
i.e. $P_0^2 = P_0$, "the exact discrete projection"
but it is preferable to use the usual discrete Laplacian,

$$\Delta_h u = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{h^2}$$

Then we get an "approximate projection" $\tilde{P}$, $\tilde{P}^2 \neq \tilde{P}$.
Cf. Almgren, Bell, Crutchfield, 2000.

$$\tilde{P} = I - \nabla_h \Delta_h^{-1} \nabla_h \cdot = \text{"approx projection"}$$
$$P_0 = I - \nabla_h \Delta_w^{-1} \nabla_h \cdot = \text{"exact discrete projection"}$$

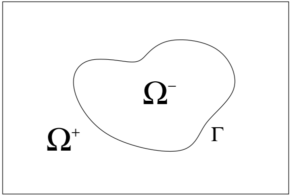
With periodic boundary conditions,
$\Delta_h = 0$ on constants, invertible on fcns of mean value zero
Δ_w has a null space with dim 2^d, $= 0$ on constants and $(-1)^j$.
Δ_w is invertible on $\perp$ to its null space

On L^2, $\|\tilde{P}\| \leq 1$, from checking eigenvalues (L^2 stability)
On L^∞, $\|\tilde{P}\| \leq C|\log h|$, using the elliptic estimate.
Also on L^∞, $\|P_0\| \leq C|\log h|$. (Grid size $2h$)
$D_h^2 \Delta_w^{-1}$ is bounded by $C|\log h|$ for centered differences!

Often with interfaces a MAC (staggered) grid is used
so that the approx proj'n is an exact proj'n.

Navier-Stokes with a moving interface

periodic bdry cond'ns on box
interface Γ, depending on t
force f on Γ acts on fluid
force on Γ amounts to jumps
in p and ∇u (here in 2D)



$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \mu \Delta u + \; f \delta_\Gamma$$

$$\nabla \cdot u = 0$$

$$[p] = f \cdot n, \quad \left[\frac{\partial p}{\partial n}\right] = \frac{\partial}{\partial s}\left(f \cdot \tan\right),$$

$$[u] = 0, \quad \mu \left[\frac{\partial u}{\partial n}\right] = -\left(f \cdot \tan\right) \tan.$$

The Scheme for Navier-Stokes

time step $\tau = O(h)$, viscosity $= 1$

$$u^{n+1} - u^n = -\tau(u \cdot \nabla u)^{n+1/2} - \tau \nabla p^{n+1/2} + \tau \Delta u^{n+1/2} + \tau C_1 + \tau C_7$$

C_1 corrects $u_t^{n+1/2} \approx (u^{n+1} - u^n)/\tau$ if crossing Γ

$$(u \cdot \nabla u)^{n+1/2} = \tfrac{3}{2} u^n \cdot \nabla_h u^n - \tfrac{1}{2} u^{n-1} \cdot \nabla_h u^{n-1} + C_2$$

$$\Delta u^{n+1/2} = \tfrac{1}{2} \left(\Delta_h u^{n+1} + \Delta_h u^n \right) + C_3$$

C_2 and C_3 correct differences using jumps in Du, $D^2 u$

$$\Delta_h p^{n+1/2} = -\left(\nabla_h \cdot (u \cdot \nabla u)^{n+1/2} + C_4 \right) + C_5 - m$$

C_5 uses jumps in p, $\partial p / \partial n$, $u \cdot \nabla u$
m is a constant so RHS has mean value zero

$$\nabla p^{n+1/2} = \nabla_h p^{n+1/2} + C_6$$

This scheme is similar to ones that have been developed:

Z. Li, M.-C. Lai, JCP 2001
L. Lee, R. LeVeque, SISC 2003
D. Le, B. Khoo, J. Peraire, JCP 2006
S. Xu, Z. J. Wang, JCP 2006

We assume periodic boundary conditions in the computational box.
With a solid boundary, the bdry cond'n must be discretized.
Discretizing at the boundary and at the interface
should be separate issues?
Often a MAC grid is used; the analysis should apply.
Xu & Wang use explicit time stepping, larger Re.

Error Estimates, Statement of Result

Theorem. Assume the exact solution is good for $0 \le t \le T$.
Neglect errors in the location of Γ.
Assume f and $\nabla_{\text{tan}} f$ are known to $O(h^2)$.
Choose time step τ with τ/h fixed. Then for $t_n = n\tau \le T$

$$\max_{j,n} \left| u^{computed}(x_j, t_n) - u^{exact}(x_j, t_n) \right| \le K_T h^2 |\log h|^2$$

Similarly the pressure error is bounded by $h^2 |\log h|^3$
except for an indefinite constant.

Error Estimate for NSE

u, p computed quantities; v, q exact

$$v^{n+1} - v^n = -\tau(v\cdot\nabla v)^{n+1/2} - \tau\nabla q^{n+1/2} + \tau\Delta v^{n+1/2} + \tau C_1 + \tau C_7 - \tau\varepsilon^n$$

Here $q^{n+1/2}$ solves a Poisson problem like $p^{n+1/2}$, v in place of u
For example, the error in $\Delta v^{n+1/2}$ is

$O(h^2)$ at regular points, $O(h)$ at irregular points

It has the form $O(h^2) + DO(h^2) = (I + D)O(h^2)$ by the Lemma
We show the truncation error is

$\varepsilon^n = (I + D)O(h^2|\log h|), \quad \varepsilon^0 = O(h|\log h|)$

Let $w = u - v =$ velocity error; subtract equations, cancel C's !
Set $g^{n+1/2} \equiv (u \cdot \nabla_h u)^{n+1/2} - (v \cdot \nabla_h v)^{n+1/2}$. Then

$$\nabla_h p^{n+1/2} - \nabla_h q^{n+1/2} = -\nabla_h \Delta_h^{-1} \nabla_h \cdot g^{n+1/2} = -(I - \tilde{P})g^{n+1/2}$$

$$w^{n+1} - w^n = -\tau \tilde{P} g^{n+1/2} + (\tau/2)(\Delta_h w^{n+1} + \Delta_h w^n) + \tau\varepsilon^n$$

Error Estimate for NSE, page 2 of 4

$w = u - v =$ velocity error; $\tilde{P} =$ approx proj'n, $P_0 =$ exact proj'n

$$w^{n+1} - w^n = -\tau \tilde{P} g^{n+1/2} + (\tau/2)(\Delta_h w^{n+1} + \Delta_h w^n) + \tau \varepsilon^n$$

$$R = (I - \tfrac{\tau}{2}\Delta_h)^{-1}, \quad S = (I + \tfrac{\tau}{2}\Delta_h)(I - \tfrac{\tau}{2}\Delta_h)^{-1}$$

$$w^{n+1} = S w^n - \tau R \tilde{P} g^{n+1/2} + \tau R \varepsilon^n$$

Since $\|\tilde{P}\| \sim |\log h|$, a direct stability estimate doesn't work?
Define $y^n = P_0 w^n$ and $z^n = (I - P_0) w^n$, and estimate separately.
$\|w^n\| \leq \|y^n\| + \|z^n\|$ but not the reverse!

$$y^{n+1} = S y^n - \tau R P_0 g^{n+1/2} + \tau P_0 R \varepsilon^n$$

$$z^{n+1} = S z^n - \tau R A (I - P_0) g^{n+1/2} + \tau (I - P_0) R \varepsilon^n$$

where $A = (\Delta_h - \Delta_w)\Delta_h^{-1}$ is bounded indep't of h
(proof uses the Fourier multiplier)

Error Estimate for NSE, page 3 of 4

$$y^{n+1} = Sy^n - \tau R P_0 g^{n+1/2} + \tau P_0 R \varepsilon^n$$

$y^n = P_0 w^n$; $g^{n+1/2} = (u \cdot \nabla_h u)^{n+1/2} - (v \cdot \nabla_h v)^{n+1/2}$
We know g is bounded like $\|D_h y\| + \|D_h z\|$.
We need to know the same for $P_0 g = g - \nabla_h \Delta_w^{-1}(\nabla_h \cdot g)$.
How to avoid the $\log h$ from P_0?

$$g = v \cdot \nabla_h w + \cdots = v \cdot \nabla_h y + v \cdot \nabla_h z + \ldots$$

For exact equations, $\nabla \cdot (v \cdot \nabla v) = \Sigma v_{j,i} v_{i,j}$ since $\nabla \cdot v = 0$.
Here $y^n = P_0 w^n$, $\nabla_h \cdot y^n = 0$, and the same works.

$z^n = (I - P_0) w^n$ is a discrete gradient, $D_{hi} z_j = D_{hj} z_i$ and
$(I - P_0)(v \cdot \nabla_h z) = \nabla_h \Delta_w^{-1} \Delta_w (v_j z_j) + D_h B(z) = \nabla_h (v_j z_j) + D_h B(z)$
All together $P_0 g^{n+1/2} = \Phi_0^n + D_h \Phi_1^n +$ (same $(n-1)$)
where $\|\Phi_k^n\| \le K(\|y^n\| + \|z^n\|)$, $k = 0$ or 1
It is important that v is continuous so that Dv is bounded!

Error Estimate for NSE, concluded

$$y^{n+1} = Sy^n - \tau RP_0 g^{n+1/2} + \tau P_0 R\varepsilon^n$$

$$y^{n+1} = -\tau \sum_{\ell=1}^{n} S^{n-\ell} RP_0 g^{\ell+1/2} + \tau \sum_{\ell=0}^{n} S^{n-\ell} RP_0 \varepsilon^{\ell}$$

We use $\|S^{n-\ell} RD_h\| \sim ((n-\ell)\tau))^{-1/2}$, $\quad P_0 g \sim Dy + Dz$
Define $\delta^n = \max\left(\|y^m\| + \|z^m\|\right)\,,\ m \le n$

$$|\text{first sum}| \sim \sum_{\ell=1}^{n-1}((n-\ell)\tau))^{-1/2}\delta^{\ell}\tau + \tau^{-1/2}\delta^n\tau$$

Similarly for the second sum, with $\varepsilon^{\ell} \sim (I + D)O(h^2|\log h|^2)$.
Add a similar estimate for z^n to get

$$\delta^{n+1} \le K_1 \sum_{\ell=1}^{n-1}((n-\ell)\tau))^{-1/2}\delta^{\ell}\tau + K_1\tau^{-1/2}\delta^n\tau + K_2 h^2(\log h)^2$$

and finally, with Hölder and Grönwall,

$$\delta^{n+1} \le Kh^2(\log h)^2$$

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