Optimal Control in a Free Boundary Fluid Elasticity Interaction

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- 3 First Order Necessary Conditions of Optimality

- Control Problem

PDE System

Configuration and Domain

- Consider an elastic body located at time $t \ge 0$ in a domain $\Omega^e(t) \subset \mathbb{R}^3$
- Let $\partial \Omega^e(t) = \Gamma(t)$
- The solid is surrounded by a fluid in \mathbb{R}^3
- Let Ω be a bounded domain comprising the two open domains Ω = Ω^e(t) ∪ Ω^f(t)
- Let Ω have smooth boundary $\partial \Omega = \Gamma^f$



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Fluid

- The fluid occupies domain $\Omega^{f}(t) = \Omega \setminus \overline{\Omega^{e}(t)}$, which is moving with time, and has smooth boundary $\Gamma(t) \cup \Gamma^{f}$
- Assume the fluid is Newtonian, viscous, and incompressible. Its behavior is described by its velocity u and pressure p
- The viscosity of the fluid is $\nu > 0$
- The fluid strain tensor is given by

$$\varepsilon(u)=\frac{Du+(Du)^*}{2}$$

The fluid state satisfies the following Navier-Stokes equations with given forcing term F:

$$\begin{cases} u_t - \nu \Delta u + (Du)u + \nabla p = F|_{\Omega^f(t)} & \text{on } \Omega^f(t) \\ \text{div } u = 0 & \text{on } \Omega^f(t) \end{cases}$$

 The fluid equations have homogeneous Dirichlet boundary conditions on the boundary Γ^f, that is,

$$u = 0$$
 on Γ^{f}

Mapping

- The evolution of the fluid domain Ω^f(t) is induced by the structural deformation through the common interface Γ(t)
- Ω^f(t) is described according to a map acting in a fixed reference domain
- The material reference configuration for the solid is $\Omega_0^e := \Omega^e(0) \subset \Omega$ with boundary $\Gamma_0 := \Gamma(0)$, and $\Omega_0^f := \Omega^f(0) = \Omega \setminus \overline{\Omega_0^e}$ is the reference fluid configuration
- The volume Ω is described by a smooth, injective map:

$$\varphi: \bar{\Omega} \times \mathbb{R}^+ \longrightarrow \bar{\Omega}, \quad (x,t) \mapsto \varphi = \varphi(x,t)$$

Deformation

- ► For $x \in \Omega_0^e$, $\varphi(x, t)$ is the position at time $t \ge 0$ of the material point x
- φ preserves the boundary Γ^f , i.e. $\varphi = I_{\Gamma^f}$ on Γ^f
- J(φ) := det(Dφ) defines the Jacobian of the deformation. We take φ to be orientation-preserving so

 $J(\varphi) > 0$

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Solid

- Assume a St. Venant-Kirchoff material, which assumes large displacement and small deformation
- Elasticity evolves according to the response function for the second Piola-Kirchoff stress tensor,

$$\Sigma = \lambda \mathsf{Tr}(\sigma(\varphi))I + \mu(\sigma(\varphi))$$

where

$$\sigma(\varphi) = \frac{1}{2}((D\varphi)^*D\varphi - I)$$

is the Green-St. Venant nonlinear strain tensor and $\lambda>0$ and $\mu>0$ are the Lamé constants of the material

Solid

 The equation of elastodynamics is written on the reference configuration Ω₀^e in terms of the response function for the first Piola-Kirchoff stress tensor

$$\mathcal{P} = D\varphi \Sigma(\sigma(\varphi)).$$

 \blacktriangleright Then φ satisfies the nonlinear elastodynamics equations:

$$\varphi_{tt} - \mathsf{Div}\mathcal{P} = 0 \text{ on } \Omega_0^e$$

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Interaction

The interaction takes place on the interface and is realized via suitable transmission boundary conditions by requiring continuity of both fluid and boundary velocities, and of the normal stress tensors across Γ :

$$\begin{cases} u \circ \varphi = \varphi_t & \text{on } \Gamma_0 \\ \mathcal{P}n = J(\varphi)(\sigma(p, u) \circ \varphi)(D\varphi)^{-*}n & \text{on } \Gamma_0, \end{cases}$$

where $\sigma(p, u) = -pl + 2\nu\varepsilon(u)$ and n = n(t) is the unit outer normal vector along $\Gamma(t)$ with respect to $\Omega^{e}(t)$.

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Control Problem

To summarize, the PDE model in the mixed Eulerian-Lagrangian formulation is

$$\begin{cases} u_t - \nu \Delta u + (Du)u + \nabla p = F & \Omega^f(t) \\ \operatorname{div} u = 0 & \Omega^f(t) \\ \varphi_{tt} - \operatorname{Div} \mathcal{P} = 0 & \Omega_0^e \\ u \circ \varphi = \varphi_t & \Gamma_0 \\ \mathcal{P}n = J(\varphi)(\sigma(p, u) \circ \varphi)(D\varphi)^{-*}n & \Gamma_0 \\ \varphi = I_{\Gamma^f} & \Gamma^f \end{cases}$$

(1)

with initial conditions

$$\varphi(\cdot,0)=\varphi^0, \varphi_t(\cdot,0)=\varphi^1, u(\cdot,0)=u^0, p(\cdot,0)=p^0 \text{ on } (\Omega_0^e)^2 \times (\Omega_0^f)^2.$$

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Control

- In many applications, a central goal is the optimization or optimal control of a considered process
 - optimize fluid velocity or pressure
 - optimize the deformation of the structure
 - minimize wall shear stresses
- ► We focus on minimization of turbulence in the fluid
- The optimization addressed from the point of view of optimal control determining the optimal action upon the system in order to minimize vorticity of the flow.

The minimization problem: for an admissible set of controls U, minimize

$\min_{F\in U}J(F)$

$$J(F) = \frac{1}{2} \int_0^T \int_{\Omega^f(t)} |\nabla \times u|^2 d\Omega^f(t) dt + \frac{1}{2} \int_0^T ||F||^2_{\mathbf{H}^s(\Omega^f(t))} dt \qquad (\mathsf{P})$$

Goals:

I Existence of optimal control acting inside the fluid domain.

This will show that the turbulence inside the flow can be reduced by applying a body force on the fluid.

2 First order necessary conditions of optimality

- Finding a suitable adjoint problem and using it to explicitly compute the gradient of the functional *J*.
- Characterization of the optimal control, paving the way for a numerical study of the problem.

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Lagrangian variables

We move the system to the Lagrangian frame in the variables v, q, and f representing the velocity, pressure, and forcing term, respectively, in reference configuration Ω_0^f :

►
$$v(x,t) = \varphi_t(x,t) = u(\varphi(x,t),t)$$
 $x \in \Omega_0^f$
► $q(x,t) = p(\varphi(x,t),t)$

•
$$f(x,t) = F(\varphi(x,t),t)$$

We require the following functional framework. Define

- ► $V_f^4(T) = \{ w \in L^2(0, T; \mathbf{H}^4(\Omega_0^f)) \mid \partial_t^n w \in L^2(0, T; \mathbf{H}^{4-n}(\Omega_0^f)), n = 1, 2, 3 \}$
- ► $V_e^4(T) = \{ w \in L^2(0, T; \mathbf{H}^4(\Omega_0^e)) \mid \partial_t^n w \in L^2(0, T; \mathbf{H}^{4-n}(\Omega_0^e)), n = 1, 2, 3 \}$

$$\blacktriangleright X_T = \left\{ v \in L^2(0, T; \mathbf{H}^1_0(\Omega)) \mid \left(v^f, \int_0^t v^e \right) \in V^4_f(T) \times V^4_e(T) \right\}$$

►
$$W_T = \{ v \in X_T \mid v_{ttt} \in \mathbf{L}^{\infty}(0, T; L^2(\Omega)), \ \partial_t^n \int_0^{\infty} v^e \in L^{\infty}(0, T; \mathbf{H}^{4-n}(\Omega_0^e)), \ n = 0, 1, 2, 3 \}$$

► $Y_T = \{ q \in L^2(0, T; \mathbf{H}^3(\Omega_0^f)) \mid \partial_t^n q \in L^2(0, T; \mathbf{H}^{3-n}(\Omega_0^f;)), n = 1, 2 \}$

$$\blacktriangleright Z_T = \{ q \in Y_T \mid q_{tt} \in L^{\infty}(0, T; \mathbf{L}^2(\Omega_0^f)) \}$$

 $\blacktriangleright E = \{ f \in L^2(0, \overline{T}; \mathbf{H}^3(\Omega) \mid (f_t, f_{tt}, f_{ttt} \in L^2(0, \overline{T}; \mathbf{H}^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)) \}.$

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Existence Result [Coutand and Shkoller (2006)]

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class H^4 . Let $\overline{T} > 0$, and let $\nu > 0$, $\lambda > 0$, $\mu > 0$ be given. Let $\overline{\Omega^e(t)} \subset \Omega$ be the closure of the open set representing the solid body at each time $t \in [0, \overline{T}]$. For $f \in E$ with $f(0) \in H^4(\Omega), f_t(0) \in H^4(\Omega)$, and necessary compatibility conditions on initial data, there exists $T \in (0, \overline{T})$ depending on u_0, f , and Ω_0^f , such that there exists a unique solution $(v, q) \in W_T \times Z_T$ of the NS-elasticity coupling.

Theorem 1 (Bociu, Castle, M., Toundykov)

Given initial data satisfying the regularity and compatibility conditions as in the existence result of Coutand and Shkoller, there exists a solution to the minimization problem (P).

That is, there exists $\overline{f} \in Q_{ad} = \overline{B_E(0,R)} = \{f \in E \mid ||f||_E \leq R\}$, for some fixed R > 0, and $(\overline{v},\overline{q}) \in W_T \times Z_T$ such that the functional J(f) attains a minimum at \overline{f} , and $(\overline{v},\overline{q})$ is the solution of the NS-elasticity coupling with forcing term \overline{f} .

Summary of the proof:

- Let $\{f_n\} \in Q_{ad}$ be a minimizing sequence for J
- ► (v_n, q_n) = (v_{f_n}, q_{f_n}), where the latter is the associated solution of the NS-elasticity coupling with right hand side f_n
- ▶ By the coercivity of J, we know that {f_n} is a bounded sequence in E. Now we use the following estimate given in [3]:

$$\|(v_n, q_n)\|_{W_T \times Z_T} \le 12C_{\delta_0} \ [N_0(u_0, (w_i)_{i=1}^3) + M_0(f_n) + N((q_i)_{i=0}^2)],$$
(*)

 $\forall n, \text{ where } C_{\delta_0} \text{ is a constant, } N_0(u_0, (w_i)_{i=1}^3) \text{ and } N((q_i)_{i=0}^2) \text{ are generic smooth functions depending only on } \\ \sum_{i=0}^3 [\|w_{3-i}\|_{H^i(\Omega_0^e)} + \|w_{3-i}\|_{H^i(\Omega_0^f)}] \text{ and } \sum_{i=0}^2 \|q_{2-i}\|_{H^i(\Omega_0^f)}, \text{ and } M_0(f_n) \text{ is a smooth function depending on } \|f_n\|_E$

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- ▶ Therefore (*) gives that (v_n, q_n) is a bounded sequence in $W_T \times Z_T$
- ► Let $W(0, T) = \{v \in L^2(0, T; H^1_0(\Omega)) | \partial_t^3 v \in L^2(0, T; L^2(\Omega))\}$ [4], which is a Hilbert space
- $W_T \subset W(0, T)$
- Let $\tilde{X}_T = X_T \cap W(0, T)$ (which is a Hilbert space)
- ► $\|(v_n, q_n)\|_{\tilde{X}_T \times Y_T} \le \|(v_n, q_n)\|_{W_T \times Z_T}$ for all $(v_n, q_n) \in W_T \times Z_T$, so $\{(v_n, q_n)\}$ is a bounded sequence in $\tilde{X}_T \times Y_T$
- ▶ Then there exists a pair $(\bar{f}, (\bar{v}, \bar{q}))$ and a subsequence, still denoted by $(f_n, (v_n, q_n))$ such that

$$\begin{cases} f_n \rightharpoonup \bar{f} & \text{in } E, \\ (v_n, q_n) \rightharpoonup (\bar{v}, \bar{q}) \in \tilde{X}_T \times Y_T, \\ \nabla \times v_n \rightharpoonup \nabla \times \bar{v} \in L^2(0, T; L^2(\Omega)) \end{cases}$$

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▶ By lower-semicontinuity, we have that

$$\|\bar{f}\|_{E}^{2} \leq \liminf_{n \to \infty} \|f_{n}\|_{E}^{2} \text{ and}$$
$$\|\nabla \times \bar{v}\|_{L^{2}(0,T;L^{2}(\Omega))} \leq \liminf_{n \to \infty} \|\nabla \times v_{n}\|_{L^{2}(0,T;L^{2}(\Omega))}$$
$$J(\bar{f}) = \|\bar{f}\|_{E}^{2} + \|\nabla \times \bar{v}\|_{L^{2}(0,T;L^{2}(\Omega))}^{2} \leq \liminf_{n \to \infty} J(f_{n})$$
$$\implies J(\bar{f}) = \liminf_{n \to \infty} J(f_{n})$$

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- Existence of Optimal Control

Existence of Optimal Pair

Desired Regularity Velocity

 $\begin{cases} \partial_{ttt} v_n \to \partial_{ttt} \bar{v} & \text{weakly in } L^2(0, T; \mathbf{L}^2(\Omega)), \\ \partial_{ttt} v_n \to \tilde{v} & \text{weakly* in } L^\infty(0, T; \mathbf{L}^2(\Omega)) \end{cases}$ For $\phi \in \mathbf{C}^{\infty}_{c}((0, T) \times \Omega)$ we have $\int_0^t \int_\Omega \partial_{ttt} v_n \cdot \phi \to \int_0^t \int_\Omega \partial_{ttt} \bar{v} \cdot \phi \text{ and}$ $\int_{0}^{T} \int_{0} \partial_{ttt} v_{n} \cdot \phi \to \int_{0}^{T} \int_{0} \tilde{v} \cdot \phi$ $\implies \tilde{v} = \partial_{ttt} \bar{v}$ a.e. in $(0, T) \times \Omega$

So

 $\partial_{ttt} \bar{v} \in L^{\infty}(0, T; \mathbf{L}^2(\Omega)).$

Similarly we obtain that

$$\partial_t^k \int_0^{\cdot} \bar{v}^e$$
 in $L^{\infty}(0, T; \mathbf{H}^{4-k}(\Omega_0^e)), k = 0, 1, 2, 3.$

• So $\bar{v} \in W_T$.

Pressure

$$\begin{cases} \partial_{tt}q_n \to \partial_{tt}\bar{q} & \text{weakly in } L^2(0, T; \mathbf{H}^1(\Omega_0^f)) \text{ and} \\ \partial_{tt}q_n \to \tilde{q} & \text{weakly* in } L^\infty(0, T; \mathbf{L}^2(\Omega_0^f)). \end{cases}$$

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• So for $\phi \in \mathbf{C}^\infty_c((0, \mathcal{T}) imes \Omega^f_0)$ we have

$$\int_0^T \int_{\Omega_0^f} \partial_{tt} q_n \cdot \phi \to \int_0^T \int_{\Omega_0^f} \partial_{tt} \bar{q} \cdot \phi$$

and

$$\int_0^T \int_{\Omega_0^f} \partial_{tt} q_n \cdot \phi \to \int_0^T \int_{\Omega_0^f} \tilde{q} \cdot \phi$$

- Thus $\tilde{q} = \partial_{tt} \bar{q}$ a.e. in $(0, T) \times \Omega_0^f$.
- Consequently the weak limit \bar{q} is in Z_T .
- ▶ Taken together, we have $(ar{
 u},ar{q})\in W_{\mathcal{T}} imes Z_{\mathcal{T}}.$

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Revisiting Goals

1 Existence of optimal control

- Showing that we can pass with the limit follows from Lions-Aubin compactness arguments due to high regularity on velocity and pressure [6].
- **2** First order necessary conditions of optimality
 - Finding a suitable adjoint problem and using it to explicitly compute the gradient of the functional *J*.
 - A linearization was obtained by Bociu and Zolésio [7].

Challenges and Ongoing Work

- The linearization shows that the boundary and its curvatures play an essential role in the final dynamical linearized systems around some equilibrium and cannot be neglected.
- Simplification and well-posedness of the adjoint system for the steady state problem.
- Finding a suitable adjoint system in the dynamical case and using it to explicitly compute the gradient of the corresponding cost functional.
- In both cases, characterizing the optimal control via the first order necessary conditions of optimality.

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