Error Analysis of Immersed Interface Method For Stationary Stokes Interface Problem

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Introduction of stationary Stokes interface problem

The incompressible Navier-Stokes equations:

$$\begin{cases} \rho\left(\dfrac{\partial u}{\partial t} + (u\cdot\nabla)u\right) - \nu\nabla^2 u = -\dfrac{\nabla p}{p_0} + g \\ \nabla\cdot u = 0 \end{cases} \tag{1}$$

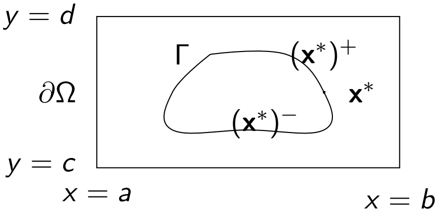
where u is velocity, p is pressure, ρ, ν are coefficients and g is force term. We assume that ν is large enough. Thus the Reynolds number will be low enough. This means that $\dfrac{\partial u}{\partial t}$ and $(u\cdot\nabla)u$ will be eliminated and the flow will be stationary.

Introduction of stationary Stokes interface problem

Bubble

Introduction of stationary Stokes interface problem

Consider the equations on a rectangular domain, we have an interface Γ in it.



We define the jump in the pressure p at point $\mathbf{x}^*$ on the interface Γ as

$$[p] = \lim_{\mathbf{x}\to(\mathbf{x}^*)^+} p - \lim_{\mathbf{x}\to(\mathbf{x}^*)^-} p \tag{2}$$

The other jumps have similar definitions.

Introduction of stationary Stokes interface problem

The stationary Stokes interface problem:

$$\begin{cases} \nabla p &= \mu \Delta u + F + \mathbf{g}, \\ \nabla \cdot u &= 0, \\ u\big|_{\partial\Omega} &= 0, \\ [p] &= 2\left[\mu \dfrac{\partial u}{\partial n} \cdot n\right] + \hat{f}_1 \\ \left[\dfrac{\partial p}{\partial n}\right] &= [\mathbf{g} \cdot n] + \dfrac{\partial}{\partial \tau}\hat{f}_2 + 2\left[\mu \dfrac{\partial^2}{\partial \tau^2}(u \cdot n)\right] \\ \left[\mu \dfrac{\partial u}{\partial n} \cdot \tau\right] &+ \left[\mu \dfrac{\partial u}{\partial \tau} \cdot n\right] + \hat{f}_2 = 0 \\ [\mu \nabla \cdot u] &= 0 \end{cases} \tag{3}$$

where F is the source along the interface $F = \int_S f(s)\delta_2(\mathbf{x} - \mathbf{X(s)})ds$, $\hat{f}_1 = f \cdot n$ and $\hat{f}_2 = f \cdot \tau$. The jump conditions are derived by Z. Li, Ito and X. Wan. [5].

Motivation

- We are using Cartesian mesh. It is the most common and trivia mesh we use in the finite difference method.
- Lots of methods can make the velocity be second order accuracy and first order accuracy for pressure. J. T. Beale has already proved that the computational solution can reach second order accuracy for elliptic interface problems.
- The Neumann boundary condition for pressure has been derived by H. Johnston and Jian-Guo Liu.
- The main idea is decoupling the Stationary Stokes interface problem into three Poisson equations and apply augmented approach to it.
- Decoupling the Stokes problem into three Poisson equations can help decouple the interface conditions.

Decouple into three Poisson equations

$$
\begin{cases}
\Delta p & = \nabla \cdot (F + \mathbf{g}) \\
[p] & = \hat{f}_1 - 2\dfrac{\partial \mathbf{q}}{\partial \tau} \cdot \tau, \left[\dfrac{\partial p}{\partial n}\right] = \dfrac{\partial \hat{f}_2}{\partial \tau} + 2\dfrac{\partial^2 (\mathbf{q} \cdot n)}{\partial \tau^2} \\
\Delta \tilde{u} & = p_x - (F_1 + g_1) \\
[\tilde{u}] & = q_1, \left[\dfrac{\partial \tilde{u}}{\partial n}\right] = \left(\hat{f}_2 + \dfrac{\partial \mathbf{q}}{\partial \tau} \cdot n\right) \sin\theta - \left(\dfrac{\partial \mathbf{q}}{\partial \tau} \cdot \tau\right) \cos\theta \\
\Delta \tilde{v} & = p_y - (F_2 + g_2) \\
[\tilde{v}] & = q_2, \left[\dfrac{\partial \tilde{v}}{\partial n}\right] = -\left(\hat{f}_2 + \dfrac{\partial \mathbf{q}}{\partial \tau} \cdot n\right) \cos\theta - \left(\dfrac{\partial \mathbf{q}}{\partial \tau} \cdot \tau\right) \sin\theta \\
\left[\dfrac{\tilde{u}}{\mu}\right] & = 0, \left[\dfrac{\tilde{v}}{\mu}\right] = 0
\end{cases}
\quad (4
$$

Decouple into three Poisson equations

where $\tilde{u} = \mu u$, $\tilde{v} = \mu v$, $\mathbf{g} = (g_1, g_2)^\intercal$,$F = (F_1, F_2)^\intercal$ and $\mathbf{q} = (q_1, q_2) = ([\tilde{u}], [\tilde{v}]) = ([\mu u], [\mu v])$ which is augment variable.

According to [11][12], we can solve out $\mathbf{q} = (q_1, q_2)$. Thus, for the pressure Poisson equation, we have

$$\begin{cases} \Delta p &= \nabla \cdot (F + \mathbf{g}) \\ [p] &= \hat{f}_1 - 2\dfrac{\partial \mathbf{q}}{\partial \tau} \cdot \tau, \left[\dfrac{\partial p}{\partial n}\right] = \dfrac{\partial \hat{f}_2}{\partial \tau} + 2\dfrac{\partial^2 (\mathbf{q} \cdot n)}{\partial \tau^2} \end{cases} \tag{5}$$

with $\mathbf{q}$ known, we can apply immersed interface method on this equations.

Neumann boundary condition for pressure

In our problem, we will have a boundary condition for pressure. Let $G = F + g$. We apply divergence on both sides of the stationary Stokes equation:

$$\nabla \cdot \nabla p = \mu \nabla \cdot \Delta u + \nabla \cdot G \tag{6}$$

with condition $\nabla \cdot u = 0$,

$$\Delta p = \nabla \cdot G \tag{7}$$

This is a Poisson equation for pressure p. Boundary condition is needed for the equation such that we can keep the transformed equation equivalent to the origin one.

Neumann boundary condition for pressure

We claim the following Neumann boundary condition[4] for pressure:

$$\left.\frac{\partial p}{\partial \mathbf{n}}\right|_{\partial\Omega} = \left[-\mu \mathbf{n} \cdot (\nabla \times \nabla \times u) + \mathbf{n} \cdot G\right]\Big|_{\partial\Omega} \tag{8}$$

This has been derived by H. Johnston and Jian-Guo Liu.

Augmented approach for pressure

We can rewrite the pressure boundary condition as:

$$\frac{\partial p}{\partial x} = -\Delta \tilde{u} - G_1, x = a \quad (9)$$

$$\frac{\partial p}{\partial x} = \Delta \tilde{u} + G_1, x = b \quad (10)$$

$$\frac{\partial p}{\partial y} = -\Delta \tilde{v} - G_2, y = c \quad (11)$$

$$\frac{\partial p}{\partial y} = \Delta \tilde{v} + G_2, y = d \quad (12)$$

We introduce the augmented boundary variables

$$\tilde{\mathbf{q}} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4)^{\mathsf{T}} = \left(\frac{\partial p}{\partial x}\bigg|_{x=a}, \frac{\partial p}{\partial x}\bigg|_{x=b}, \frac{\partial p}{\partial y}\bigg|_{y=c}, \frac{\partial p}{\partial y}\bigg|_{y=d}\right)^{\mathsf{T}} \quad (13)$$

Augmented approach for pressure

According to [11], we can solve $\mathbf{q}$ and $\tilde{\mathbf{q}}$ with linear system by GMRES iterative method. Our pressure Poisson equation becomes

$$\begin{cases} \Delta p & = \nabla \cdot G \\ [p] & = \hat{f}_1 - 2\dfrac{\partial \mathbf{q}}{\partial \tau} \cdot \tau \\ \left[\dfrac{\partial p}{\partial n}\right] & = \dfrac{\partial \hat{f}_2}{\partial \tau} + 2\dfrac{\partial^2 (\mathbf{q} \cdot n)}{\partial \tau^2} \\ \dfrac{\partial p}{\partial n} & = \tilde{\mathbf{q}} \end{cases} \tag{14}$$

Then we will consider the immersed interface method on this equations. Set $\nabla \cdot G = F$. (The F here is different from the one representing source along interface)

Goal

Many methods to Stokes equation can reach second order accuracy on velocity but first order accuracy on pressure. There are some computational examples such as Z. Li's Stokes-Darcy Fluid Structure showing that the approach we introduced above can reach second order accuracy on both velocity and pressure. My goal is proving theoretically that the approach can reach second order on pressure in stationary Stokes equation even though pressure has first order accuracy around the interface.

Notation

Without loss of generality, we set the rectangular starting from origin. Some notations[1]:

- rectangular region $\Omega = \{x \in R^2 : 0 < x < Nh, 0 < y < Mh\}$
- computation domain $\Omega_h = \{jh \in hZ^2 : 1 \le j_1 \le N-1, 1 \le j_2 \le M-1\}$
- boundary $\partial\Omega_h = \{jh : 0 \le j_1 \le N, j_2 = 0 \text{ or } M; 0 \le j_2 \le M, j_1 = 0 \text{ or } N\}$
- second-order discrete Laplacian $\Delta_h p^h = D_1^- D_1^+ p + D_2^- D_2^+ p$

Difference scheme on the boundary

- On the boundary, the difference scheme

$$d_1 p(0, j_2 h) = \frac{p(h, j_2 h) - p(-h, j_2 h)}{2h} \quad (15)$$

$$d_1 p(Nh, j_2 h) = \frac{p((N+1)h, j_2 h) - p((N-1)h, j_2 h)}{2h} \quad (16)$$

$$d_2 p(j_1 h, 0) = \frac{p(j_1 h, h) - p(j_1 h, -h)}{2h} \quad (17)$$

$$d_2 p(j_1 h, Mh) = \frac{p(j_1 h, (M+1)h) - p(j_1 h, (M-1)h)}{2h} \quad (18)$$

Let p^e be the exact solution. At regular points, the truncation error is

$$\Delta_h p^e(jh) = F_\pm(jh) + \tau^h(jh), \quad |\tau^h(jh)| \le Ch^2 \tag{19}$$

At irregular point, there is $T^h(jh)$, determined by jumps on Γ.

$$\Delta_h p^e(jh) = F_\pm(jh) + T^h(jh) + \tau^h(jh), |\tau^h(jh)| \le Ch \tag{20}$$

Define F^h on Ω_h

$$F^h(jh) = \begin{cases} F_\pm(jh) + T^h(jh), & jh \quad \text{irregular} \\ F_\pm(jh), & jh \quad \text{regular} \end{cases} \tag{21}$$

Thus p^h is the solution to

$$\Delta_h p^h = F^h \quad \text{in} \quad \Omega_h, \quad dp^h = 0 \quad \text{on} \quad \partial\Omega_h \tag{22}$$

Then the error $p^h - p^e$ will satisfy

$$\begin{aligned} \Delta_h(p^h - p^e) &= -\tau^h \quad \text{in} \quad \Omega_h, && (23) \\ d(p^h - p^e) &= -O(h^2) \quad \text{on} \quad \partial\Omega_h && (24) \end{aligned}$$

According to the Lemma 2.2 from Beale's[1]. There exist functions F_1 and F_2 such that

$$F^{irr} = D_1^- F_1 + D_2^- F_2 \quad \text{in} \quad \Omega_h \tag{25}$$

The v we are going to use is different from the one we used previous. It will represent a general function.

Lemma 1

Lemma 1

Suppose

$$\Delta_h v = F^{reg} + D_1^- F_1 + D_2^- F_2 \quad in \quad \Omega_h, \tag{26}$$

$$dv = -O(h^2) \quad on \quad \partial\Omega_h \tag{27}$$

Then

$$\|\nabla_h^+ v\|_{\overline{\Omega}_h}^2 \leq \|F^{reg}\|_{\overline{\Omega}_h}\|v\|_{\overline{\Omega}_h} + (\|F_1\|_{\overline{\Omega}_h} + O(h^3))\|D_1^+ v\|_{\overline{\Omega}_h} + (\|F_2\|_{\overline{\Omega}_h} + O(h^3))\|D_2^+ v\|_{\overline{\Omega}_h} + 4O(h^2)\|v\|_{\overline{\Omega}_h} \tag{28}$$

Lemma 1

Proof

$$(\Delta_h v, v)_{\overline{\Omega}_h} = -(\nabla_h^+ v, \nabla_h^+ v)_{\overline{\Omega}_h} + \sum_{j_2=0}^{M} [D_1^+ v(Nh, j_2 h)]^2 \cdot h^2$$

$$+ \sum_{j_1=0}^{N} [D_1^+ v(j_1 h, Mh)]^2 \cdot h^2 - \sum_{j_2=0}^{M} D_1^+ v(0, j_2 h) \cdot v(0, j_2 h) \cdot h$$

$$- \sum_{j_1=0}^{N} D_2^+ v(j_1 h, 0) \cdot v(j_1 h, 0) \cdot h + \sum_{j_2=0}^{M} D_1^+ v(Nh, j_2 h) \cdot v(Nh, j_2 h) \cdot$$

$$+ \sum_{j_1=0}^{N} D_2^+ v(j_1 h, Mh) \cdot v(j_1 h, Mh) \cdot h \qquad (29)$$

Lemma 1

Proof

For the right hand side,

$$(F^{reg} + D_1^- F_1 + D_2^- F_2, v)_{\overline{\Omega}_h} = (F^{reg}, v)_{\overline{\Omega}_h} + (D_1^- F_1, v)_{\overline{\Omega}_h} + (D_2^- F_2, v)_{\overline{\Omega}_h} \tag{30}$$

$$(D_1^- F_1, v)_{\overline{\Omega}_h} = -(F_1, D_1^+ v)_{\overline{\Omega}_h} \tag{31}$$

$$(D_2^- F_2, v)_{\overline{\Omega}_h} = -(F_2, D_2^+ v)_{\overline{\Omega}_h} \tag{32}$$

So

$$(F^{reg} + D_1^- F_1 + D_2^- F_2, v)_{\overline{\Omega}_h} = (F^{reg}, v)_{\overline{\Omega}_h} - (F_1, D_1^+ v)_{\overline{\Omega}_h} - (F_2, D_2^+ v)_{\overline{\Omega}_h} \tag{33}$$

Lemma 1

Proof

Combining $(\Delta_h v, v)_{\overline{\Omega}_h}$ *with* $(F^{reg} + D_1^- F_1 + D_2^- F_2, v)_{\overline{\Omega}_h}$, *we know on the boundary*

$$D_1^+ v = -O(h^2), D_2^+ v = -O(h^2) \tag{34}$$

With Cauchy-Schwarz inequality,

$$\begin{aligned}\|\nabla_h^+ v\|_{\overline{\Omega}_h}^2 \leq \|F^{reg}\|_{\overline{\Omega}_h}\|v\|_{\overline{\Omega}_h} + (\|F_1\|_{\overline{\Omega}_h} + O(h^3))\|D_1^+ v\|_{\overline{\Omega}_h} \\ + (\|F_2\|_{\overline{\Omega}_h} + O(h^3))\|D_2^+ v\|_{\overline{\Omega}_h} + 4O(h^2)\|v\|_{\overline{\Omega}_h}\end{aligned} \tag{35}$$

Lemma 2

Next we need Discrete Neumann-Poincare inequality:

Lemma 2

If v is C^1 on Ω_h, we define

$$A = \frac{\sum\limits_{\overline{\Omega}_h} v}{M \cdot N} \tag{36}$$

Then,

$$\|v - A\|_{\overline{\Omega}_h} \leq C \|\nabla_h^+ v\|_{\overline{\Omega}_h} \tag{37}$$

Lemma 2

Proof

In two dimension, we have a point $(c_1, c_2) \in \Omega$ *such that*

$$A = \frac{\sum\limits_{\overline{\Omega}_h} v}{M \cdot N} = v(c_1, c_2) \tag{38}$$

For any point $(x, y) \in \Omega_h$

$$v(x, y) = v(c_1, c_2) + \sum_{x'=c_1}^{x} D_1^+ v(x', c_2) h + \sum_{y'=c_2}^{y} D_2^+ v(c_1, y') h \tag{39}$$

Lemma 2

Proof

Taking the norm on Ω_h

$$\begin{aligned}\|v-A\|_{\overline{\Omega}_h} &\le \sum_{x'=0}^{Nh} \|D_1^+ v(x',c_2)h\|_{\overline{\Omega}_h} + \sum_{y'=0}^{Mh} \|D_2^+ v(c_1,y')h\|_{\overline{\Omega}_h} \\ &\le \frac{N^2h^2+M^2h^2}{2}\|\nabla_h^+ v\|_{\overline{\Omega}_h} \\ &= C\|\nabla_h^+ v\|_{\overline{\Omega}_h}\end{aligned} \tag{40}$$

Main result

Theorem

Let p^e be the exact solution of the pressure part problem with the interface Γ at least C^1. Suppose $\Delta_h p^e$ has the form given by $D_1^- D_1^+ p^e + D_2^- D_2^+ p^e$, with $\|\tau_h(jh)\| \leq Ch$ at irregular grid points and $\|\tau_h(jh)\| \leq Ch^2$ at regular grid points. Let p^h be the solution of (31),(32). And there is a chosen grid point (α, β), on which $p^h - p^e = 0$. Then,

$$\|p^h(jh) - p^e(jh)\|_{\overline{\Omega}_h} \leq C_0 h^2 \tag{41}$$

Main result

Proof

Let $v = p^h - p^e$*, then we have a chosen point*(α, β) *where* $v(\alpha, \beta) = A = 0$*. With Lemma 2*

$$\|v\|_{\overline{\Omega}_h} \leq C\|\nabla_h^+ v\|_{\overline{\Omega}_h} \tag{42}$$

Combining with the result we get from Lemma 1

$$\|\nabla_h^+ v\|^2_{\overline{\Omega}_h} \leq C\|F^{reg}\|_{\overline{\Omega}_h}\|\nabla_h^+ v\|_{\overline{\Omega}_h} + (\|F_1\|_{\overline{\Omega}_h} + O(h^3))\|\nabla_h^+ v\|_{\overline{\Omega}_h} + (\|F_2\|_{\overline{\Omega}_h} + O(h^3))\|\nabla_h^+ v\|_{\overline{\Omega}_h} + 4CO(h^2)\|\nabla_h^+ v\|_{\overline{\Omega}_h} \tag{43}$$

Main result

Proof

Divided $\|\nabla_h^+ v\|_{\overline{\Omega}_h}$ *on the both sides and use the result from Lemma 2.2 in Beale's[1]*

$$\|F_k\|_{\overline{\Omega}_h} = O(h^2), \|F^{reg}\|_{\overline{\Omega}_h} = O(h^2) \tag{44}$$

$$\|v\|_{\overline{\Omega}_h} \leq (5C+2)O(h^2) + 2O(h^3) \tag{45}$$

Thus,

$$\|p^h(jh) - p^e(jh)\|_{\overline{\Omega}_h} \leq C_0 h^2 \tag{46}$$

Conclusion and Future work

Conclusion: We can have second order accuracy computational solution to pressure by decoupling the stationary stokes equation and applying augment approach to it.

Future work:

- Elliptic interface problems with Neumann boundary condition.
- How the result will be in 3 dimensional problem?
- Splitting approach.

Thank you for your patience!

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