Integrability of moduli spaces of parahoric Higgs bundles

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August 2016
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Setup

$X =$ connected smooth projective algebraic curve over $\mathbb{C}$, genus $g \geq 2$

$\Omega =$ cotangent bundle of X.

$G =$ semisimple algebraic group over $\mathbb{C}$ (reductive is ok, but requires some modifications)

We'll use stacks in this talk. For the most part you could instead use semistable moduli spaces. There is at least one place where stacks are advantageous, however.

The Hitchin system

Let $Bun_G =$ moduli stack of principal G-bundles on X.

Hitchin found a collection $h_1, \ldots, h_d$ of Poisson commuting functions on $T^* Bun_G$. "Somewhat miraculously" the number of functions he found was exactly equal to $\dim(Bun_G)$.

With these Poisson commuting functions, $T^* Bun_G$ becomes an *algebraically completely integrable system*. In particular:

- $h = (h_1, \ldots, h_d) : T^* Bun_G \to \mathbb{C}^d$ is a Lagrangian fibration,
- h is flat and surjective
- generic fibres are abelian varieties (with a trivial gerbe)

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Generalisations

A principal G-bundle on X is a torsor for the constant group scheme $X \times G \to X$ over X.

Consider the following generalisation:
Let $\mathcal{G}$ be a group scheme over X and $Bun_{\mathcal{G}}$ the stack of $\mathcal{G}$-torsors.

(1) Can we define a "Hitchin map" on $T^*Bun_{\mathcal{G}}$?
(2) Do we get an integrable system ?

(1) can be done in many cases. In general, the answer to (2) is **no**, at least not with the obvious analogue of the Hitchin map. Counterexample on next slide.

However, if $\mathcal{G}$ is a **parahoric Bruhat-Tits group scheme** over X, the answer to (2) is **yes**, at least in some cases (and conjecturally always).

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Counterexample

Fix a marked point $x \in X$. Consider $Bun_{SL_2,x}$ the stack of principal SL_2-bundles with a framing over x.

$$\dim(Bun_{SL_2,x}) = 3g.$$

$T^*Bun_{SL_2,x}$ is the moduli stack of pairs $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a principal SL_2 bundle with framing over x and $\phi \in H^0(X, ad(\mathcal{E}) \otimes \Omega(x))$.

$$\dim(T^*Bun_{SL_2,x}) = 6g \quad (Bun_{SL_2,x} \text{ is } \textbf{good}).$$

Hitchin map:

$$h : T^*Bun_{SL_2,x} \to A_{SL_2,x} := H^0(X, \Omega^2(2x)), \quad (\mathcal{E}, \phi) \mapsto \det(\phi)$$

But $\dim(A_{SL_2,x}) = 3g - 1 < \dim(Bun_{SL_2,x}) = 3g$.

$h : T^*Bun_{SL_2,x} \to A_{SL_2,x}$ is a coisotropic fibration, but not a Lagrangian fibration. The Nilpotent cone (dimension $= 3g + 1$) is not Lagrangian!

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Parahoric and parabolic

Now we consider the case where $\mathcal{G}$ is a parahoric Bruhat-Tits group scheme over X.

To simplify matters, we will mostly discuss only the special case of parabolic bundles. Some of our results also hold in the parahoric case.

Parabolics correspond to subsets of the Dynkin diagram. Parahorics to subsets of the extended Dynkin diagram.

As a further simplification, we consider only one marked point $x \in X$. The extension to several marked points is straightforward.

Parabolic bundles

Suppose $P \subset G$ is a parabolic subgroup of G.

Let $Bun_{G,P}$ be the stack of principal G-bundles on X with reduction of structure to P over the marked point $x \in X$.

We call a principal G-bundle with such a reduction a **(quasi-)parabolic bundle**.

We will omit the prefix "quasi-", since we will not be concerned with parabolics weights or semistability.

Parabolic Higgs bundles

$Bun_{G,P}$ is a smooth equidimensional algebraic stack.

What is $T^*Bun_{G,P}$?

For $E \in Bun_{G,P}$, we have an exact sequence:

$$0 \to \mathcal{O}(ad_P(E)) \to \mathcal{O}(ad(E)) \to \mathcal{O}_x \otimes (\mathfrak{g}/\mathfrak{p}) \to 0,$$

where $\mathcal{O}(ad_P(E))$ is the sheaf of sections of $ad(E)$ preserving the reduction to P.

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Parabolic Higgs bundles

By Serre duality:

$$T^*_E Bun_{G,P} \cong H^1(X, ad_P(E))^* \cong H^0(X, ad_P(E)^* \otimes \Omega).$$

From the exact sequence defining $ad_P(E)$, we obtain a dual exact sequence:

$$0 \longrightarrow \mathcal{O}(ad(E))^* \longrightarrow \mathcal{O}(ad_P(E)^*) \xrightarrow{Res_x} (\mathfrak{g}/\mathfrak{p})^* \longrightarrow 0.$$

Using the Killing form κ to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ gives $(\mathfrak{g}/\mathfrak{p})^* = \mathfrak{n}$, the nilradical of $\mathfrak{p}$. Then:

$$T^*_E Bun_{G,P} \cong \{\phi \in H^0(X, ad(E) \otimes \Omega(x)) \mid Res_x(\phi) \in \mathfrak{n}\}.$$

We call a pair $(E, \phi) \in T^* Bun_{G,P}$ a **parabolic Higgs bundle**

(these are sometimes called “strongly parabolic Higgs bundles”)

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Parabolic Hitchin map

Let $p_1, \ldots, p_l$ be homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$ of degrees $d_1 \leq d_2 \leq \cdots \leq d_l$. If ϕ has a first order pole at x, then $p_j(\phi)$ will have at most a pole of order j.

We define the parabolic Hitchin map:

$$h : T^*Bun_{G,P} \to A^{big}_{G,P} := \bigoplus_{j=1}^{l} H^0(X, \Omega^{d_j}(d_j x))$$

$$(E, \phi) \mapsto h(E, \phi) = (p_1(\phi), \ldots, p_l(\phi)).$$

Call $A^{big}_{G,P}$ the big Hitchin base. It is an affine space of dimension d say.

A map h to $A_{G,P}$ is just a collection $(h_1, \ldots, h_d)$ of d functions on $T^*Bun_{G,P}$.

Parabolic Hitchin base

The base

$$A_{G,P}^{big} = \bigoplus_{j=1}^{l} H^0(X, \Omega^{d_j}(d_j x))$$

is much too big - the Hitchin map will take values in some proper subvariety.

This is easy to see: the reside of ϕ is nilpotent, so for any invariant polynomial p_j of degree d_j, $p_j(\phi)$ will have a pole of order strictly less than d_j.

Definition

We define the **parabolic Hitchin base** $A_{G,P}$ to be the Zariski closure in $A_{G,P}^{big}$ of the image of h.

Parabolic Hitchin base

By definition of $A_{G,P}$ we have the Hitchin map

$$h : T^*Bun_{G,P} \to A_{G,P}, \quad (E, \phi) \mapsto (p_1(\phi), \ldots, p_l(\phi))$$

and the image of h is dense in $A_{G,P}$.

What does $A_{G,P}$ look like?

This is a surprisingly difficult problem to answer.

We can pose the same problem for parahoric Higgs bundles.

Example - Borel subgroup

The easiest case is when $P = B$ is a Borel. Then:

$$A_{G,P} = \bigoplus_{j=1}^{l} H^0(X, \Omega^{d_j}((d_j - 1)x)).$$

Proof: the inclusion $\subseteq$ is obvious.

For the inclusion $\supseteq$, use a variation of the usual Hitchin-Kostant-Rallis section:

$$\phi = t^{-1}e + \sum_{j=1}^{l} a_i f_i,$$

where t is a section of $\mathcal{O}(x)$ vanishing at x and $a_j \in H^0(X, \Omega^{d_j}((d_j - 1)x))$.

Second example

Let $P \subset SL_n$ be the stabilizer of a 1-dimensional subspace $\mathbb{C}v \subset E_x$.

So ϕ has form:

$$\phi = \begin{bmatrix} \mathcal{O} & t^{-1}\mathcal{O} & t^{-1}\mathcal{O} & \cdots & t^{-1}\mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix}.$$

Then $\lambda I - \phi$ has only one row with poles in it, so $det(\lambda I - \phi)$ has only first order poles. Thus if we choose our invariant polynomials $p_1, \ldots, p_l$ to be the coefficients of the characteristic polynomial, we get:

$$A_{G,P} = \bigoplus_{j=1}^{l} H^0(X, \Omega^{d_j}(x)).$$

Second example - continued

Note that if we had used a different set of invariant polynomials, we wouldn't have got such a nice description for $A_{G,P}$.

If we have used, say, $p_j(\phi) = Tr(\phi^j)$, then $p_j(\phi)$ would pick up higher order poles. For instance (in $\mathfrak{sl}_n$):

$$Tr(\phi^4) = 2e_2^2(\phi) - 4e_4(\phi),$$

where $e_j = j$-th elementary symmetric polynomial. So $Tr(\phi^4)$ can have a second order pole.

Why the elementary symmetric functions should work better than any other basis is still mysterious to us (may be related to work of Kottwitz on Newton strata).

Conjecture

Conjecture (Parabolic miracle)

$A_{G,P}$ is an **affine space** *of dimension* $\dim(Bun_{G,P}) = \dim(Bun_G) + \mathfrak{n}$

We have an analogous conjecture in the parahoric setting.

Theorem (B.-Kamgarpour-Varma)

- *The conjecture is true for all parahorics of type A*
- *The conjecture is true for all* **parabolics** *of type A, B, C, G_2 and some parabolics of type D*
- *The conjecture is true for all parabolics of Borel type*
- *We have proven that $\dim(A_{G,P}) = \dim(Bun_{G,P})$, i.e. the dimension part of the conjecture is true for all parabolics.*

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Poisson commutativity

Recall the Hitchin map h is given by a collection of functions $h_1, \dots, h_d$.

Theorem (B.-Kamgarpour-Varma)

The functions $h_1, \dots, h_d$ are Poisson commuting.

Note: $Bun_{G,P}$ is **very good** in the sense of Beilinson-Drinfeld.

This means $T^*Bun^0_{G,P}$ is dense in $T^*Bun_{G,P}$, where $Bun^0_{G,P}$ is the biggest Deligne-Mumford substack of $Bun_{G,P}$.

This makes it easy to even say what the Poisson bracket is without entering the derived world.

Sketch proof

Let $(E, \phi) \in T^*Bun^0_{G,P}$. Consider the two-term complex:

$$\mathcal{O}(ad_P(E)) \xrightarrow{[\phi,\ ]} \mathcal{O}(ad_P(E)^* \otimes \Omega)$$

Call this complex $[\phi,\ ]$. The tangent space to (E, ϕ) is $\mathbb{H}^1(X, [\phi,\ ])$.

Serre duality (extended to hypercohomology) gives:

$$\mathbb{H}^j(X, [\phi,\ ]) \cong \mathbb{H}^{2-j}(X, [\phi,\ ]^* \otimes \Omega).$$

But $[\phi,\ ]^* \otimes \Omega \cong [\phi,\ ]$. This gives the symplectic form on $\mathbb{H}^1$.

We work out the Hamiltonian vector fields $X_1, \ldots, X_d$ associated to $h_1, \ldots, h_d$. Then check $X_1, \ldots, X_d$ span an isotropic subspace.

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Global nilpotent cone

$Nilp_{G,P} := h^{-1}(0)$ is the substack of $T^*Bun_{G,P}$ of nilpotent parabolic Higgs bundles.

Theorem (B.-Kamgarpour-Varma)

$Nilp_{G,P}$ is isotropic.

Sketch proof:

Balaji-Seshadri $\Rightarrow$ exists a Galois cover $p : Y \to X$, such that parabolic bundles on X correspond to certain equivariant principal bundles on Y.

Then follow proof of Ginzburg that $Nilp$ is isotropic for principal bundles.

Note: this proof also works for parahorics.

$Nilp_{G,P}$ is Lagrangian

1) h is Poisson commuting $\Rightarrow$ generic fibres are **coisotropic**, so:

$$\dim(Bun_{G,P}) \leq \dim(\text{generic fibre})$$

2) using the $\mathbb{C}^*$-action one can show (cf. Ginzburg):

$$\dim(\text{generic fibre}) \leq \dim(Nilp_{G,P})$$

3) $Nilp_{G,P}$ is **isotropic** gives:

$$\dim(Nilp_{G,P}) \leq \dim(Bun_{G,P})$$

1) + 2) + 3) $\Rightarrow$ equalities throughout. In particular $Nilp_{G,P}$ is Lagrangian. One can further show that every irreducible component of every fibre has same dimension.

The equality $\dim(A_{G,P}) = \dim(Bun_{G,P})$

The stack $Bun_{G,P}$ is very good, which implies good, so:

$$\dim(M_{G,P}) = 2\dim(Bun_{G,P}).$$

We have just shown that all fibres have dimension $\dim(Bun_{G,P})$. Thus

$$\dim(A_{G,P}) = \dim(M_{G,P}) - \dim(Bun_{G,P}) = \dim(Bun_{G,P}).$$

Integrability

We have shown that the parabolic Hitchin map $h : M_{G,P} \to A_{G,P}$ is a Lagrangian fibration and, by definition of $A_{G,P}$, the image of h is dense.

Theorem (B.-Kamgarpour-Varma)

Suppose the conjecture holds (i.e. $A_{G,P}$ is an affine space). Then h is flat and surjective. In particular, this holds for all parabolics in types A, B, C, G_2.

Proof: 1) $Bun_{G,P}$ good implies $T^*Bun_{G,P}$ is a local complete intersection, so Cohen-Macaulay.

2) Fibres of h all have same dimension.
By miracle flatness, 1) + 2) $\Rightarrow$ h is flat.

h flat $\Rightarrow$ image h open.

Image of h is open, $\mathbb{C}^*$-invariant and contains $0 \Rightarrow h$ surjective.

Parabolic Hitchin base - types A, B, C, G_2

To simplify things, assume $\mathfrak{g}$ is of type A, B, C or G_2. Type D works for some parabolics, but the results are messier to state.

We use the following bases of invariant polynomials:

$$\begin{aligned}
A_{n-1}: &\quad \det(\lambda - \phi) = \lambda^n + c_2\lambda^{n-2} + \cdots + c_n, \text{ use } (c_2, \ldots, c_n)\\
B_n: &\quad \det(\lambda - \phi) = \lambda^{2n+1} + c_2\lambda^{2n-1} + \cdots + c_{2n}\lambda, \text{ use } (c_2, c_4, \ldots, c_{2n})\\
C_n: &\quad \det(\lambda - \phi) = \lambda^{2n} + c_2\lambda^{2n-2} + \cdots + c_{2n}, \text{ use } (c_2, c_4, \ldots, c_{2n})\\
G_2: &\quad \det(\lambda - \phi) = \lambda^7 + c_2\lambda^5 + \cdots + c_6\lambda, \text{ use } (c_2, c_6).
\end{aligned}$$

Parabolic Hitchin base - types A, B, C, G_2

Let $L \subset P$ be the Levi subgroup, $\mathfrak{l} \subset \mathfrak{p}$ the Levi subalgebra.

$\mathfrak{l}$ is reductive of rank l, so $\mathbb{C}[\mathfrak{l}]^L$ is generated by polynomials of degrees $m_1 \leq m_2 \leq \cdots \leq m_l$, say.

Theorem (B.-Kamgarpour-Varma)

In types A, B, C, G_2 using the invariant polynomials just described we have:

$$A_{G,P} = \bigoplus_{j=1}^{l} H^0(X, \Omega^{d_j}((d_j - m_j)x)).$$

This is clearly an affine space. We will check it has the right dimension. This formula works for some (but not all!) parabolics of type D.

The equality $\dim(A_{G,P}) = \dim(Bun_{G,P})$ revisited

Note first that:

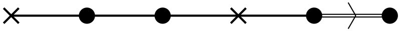
$$\begin{aligned}\dim(Bun_{G,P}) &= \dim(Bun_G) + \dim(G/P)\\ &= (g-1)\dim(G) + \dim(G/P).\end{aligned}$$

Now we calculate:

$$\begin{aligned}\dim(A_{G,P}) &= \sum_{j=1}^{l} \dim(H^0(X, \Omega^{d_j}((d_j - m_j)x)))\\ &= \sum_{j=1}^{l} (2d_j - 1)(g-1) + (d_j - m_j)\\ &= \sum_{j=1}^{l} (2d_j - 1)(g-1) + \frac{1}{2}\left((2d_j - 1) - (2m_j - 1)\right)\\ &= (g-1)\dim(G) + \frac{1}{2}\left(\dim(\mathfrak{g}) - \dim(\mathfrak{l})\right)\\ &= (g-1)\dim(G) + \dim(G/P).\end{aligned}$$

Example

We get a parabolic subgroup $P \subset G$ by crossing off nodes on the Dynkin diagram of G, eg:



We have $\mathfrak{g} = \mathfrak{so}_{13}, \quad (d_1, d_2, d_3, d_4, d_5, d_6) = (2, 4, 6, 8, 10, 12)$

The Levi $\mathfrak{l}$ is the product of the diagram obtained by removing crosses, plus a $\mathfrak{gl}_1$ for each cross:

$\mathfrak{l} = \mathfrak{sl}_3 \times \mathfrak{so}_5 \times \mathfrak{gl}_1 \times \mathfrak{gl}_1, \quad (m_1, m_2, m_3, m_4, m_5, m_6) = (1, 1, 2, 2, 3, 4).$

$$\begin{aligned} A_{G,P} = {} & H^0(X, \Omega^2(x)) \oplus H^0(X, \Omega^4(3x)) \oplus H^0(X, \Omega^6(4x)) \\ & \oplus H^0(X, \Omega^8(6x)) \oplus H^0(X, \Omega^{10}(7x)) \oplus H^0(X, \Omega^{12}(8x)). \end{aligned}$$

Sanity check: $1 + 3 + 4 + 6 + 7 + 8 = 29 = \dim(\mathfrak{n}) = \dim(G/P)$.

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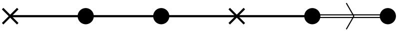

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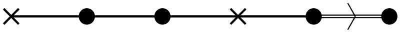
Example

We get a parabolic subgroup $P \subset G$ by crossing off nodes on the Dynkin diagram of G, eg:



We have $\mathfrak{g} = \mathfrak{so}_{13}$, $\quad (d_1, d_2, d_3, d_4, d_5, d_6) = (2, 4, 6, 8, 10, 12)$

The Levi $\mathfrak{l}$ is the product of the diagram obtained by removing crosses, plus a $\mathfrak{gl}_1$ for each cross:

$\mathfrak{l} = \mathfrak{sl}_3 \times \mathfrak{so}_5 \times \mathfrak{gl}_1 \times \mathfrak{gl}_1, \quad (m_1, m_2, m_3, m_4, m_5, m_6) = (1, 1, 2, 2, 3, 4)$.

$$\begin{aligned} A_{G,P} = {} & H^0(X, \Omega^2(x)) \oplus H^0(X, \Omega^4(3x)) \oplus H^0(X, \Omega^6(4x)) \\ & \oplus H^0(X, \Omega^8(6x)) \oplus H^0(X, \Omega^{10}(7x)) \oplus H^0(X, \Omega^{12}(8x)). \end{aligned}$$

Sanity check: $1 + 3 + 4 + 6 + 7 + 8 = 29 = \dim(\mathfrak{n}) = \dim(G/P)$.

Type A

Let $\mathfrak{g} = \mathfrak{gl}_n$ ($\mathfrak{sl}_n$ case is similar). Let $V = \mathbb{C}^n$ the standard representation, $\mathfrak{p} \subseteq \mathfrak{g}$ a parabolic. $\mathfrak{p}$ is the subalgebra preserving a flag

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k = V.$$

An endomorphism $T : V \to V$ is in $\mathfrak{p}$ if and only if $T(F_j) \subseteq F_j$ for all j.

Let $n_j = \dim(F_j/F_{j-1})$, so $n = n_1 + n_2 + \cdots + n_k$ is a partition of n.

Choose splittings $F_j = F_{j-1} \oplus V_j$, so

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k.$$

Then

$$\mathfrak{p} = \bigoplus_{i \geq j} Hom(V_i, V_j),$$

$$\mathfrak{n} = \bigoplus_{i > j} Hom(V_i, V_j).$$

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Type A

Let $\mathfrak{l}$ be the Levi of $\mathfrak{p}$:

$$\mathfrak{l} = \bigoplus_i End(V_i).$$

Let $m_0 \leq m_1 \leq \cdots \leq m_{n-1}$ denote the fundamental degrees of $\mathfrak{l}$. Let $n_1^c \geq n_2^c \geq \cdots \geq n_m^c$ be the conjugate partition of $(n_1, n_2, \ldots, n_k)$.

Since $End(V_i)$ has fundamental degrees $1, 2, \ldots, n_i$ one sees that $m_0, m_1, \ldots, m_{n-1}$ are given by:

$$\underbrace{1, 1, \ldots, 1}_{n_1^c \text{ times}}, \underbrace{2, 2, \ldots, 2}_{n_2^c \text{ times}}, \ldots, \underbrace{m, m, \ldots, m}_{n_m^c \text{ times}}.$$

Type A spectral data

The generic fibres of h are abelian varieties. If so, we should be able to say what these abelian varieties are.

Consider the GL_n case.

Let

$$a = (a_1, a_2, a_3, \ldots, a_n) \in A_{GL_n,P} = \bigoplus_{j=1}^{n} H^0(X, \Omega^j((j - m_{j-1})x)).$$

Associated to a is a spectral curve S_a in the total space of $\Omega(x)$. However, the spectral curves are all singular (unless P is a Borel).

Bertini's theorem $\Rightarrow$ the generic S_a has only one singular point (which lies over x)

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Singularities of the spectral curves

The singularity of S_a is a plane curve singularity, so it has a Newton polygon.

Theorem (Scheinost-Schottenloher, B.-Kamgarpour-Varma)

For all sufficiently generic $a \in A_{GL_n,P}$, one has:

- *The Newton polygon has slopes $-n_1^c, -n_2^c, \ldots, -n_m^c$.*
- *The normalisation $\widetilde{S_a}$ of S_a has m local branches around x*
- *$n_1^c, \ldots, n_m^c$ are the covering degrees of the restriction of $\widetilde{S_a} \to X$ to the local branches*
- *If ϕ is a parabolic Higgs field with characteristic polynomial given by a, then $Res_x(\phi)$ has m Jordan blocks of sizes $n_1^c, \ldots, n_m^c$ (i.e. $Res_x(\phi)$ is in the Richardson orbit associated to $\mathfrak{p}$)*
- *The fibre of the GL_n-parabolic Hitchin map over a is $Jac(\widetilde{S_a})$*

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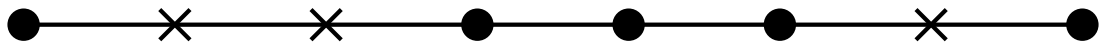
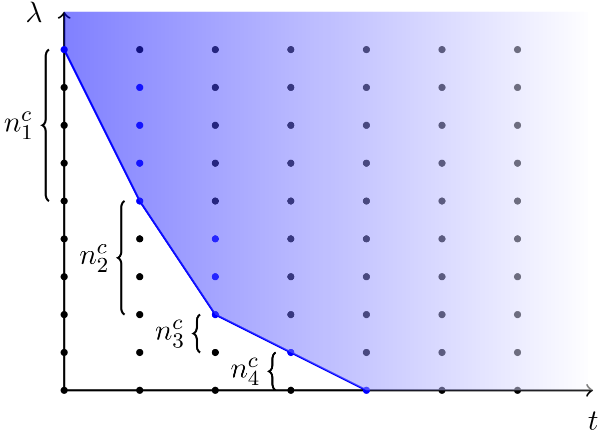
An example

eg, say $\mathfrak{p} \subseteq \mathfrak{sl}_9$ given by:



Then $(n_1, n_2, n_3, n_4) = (2, 1, 4, 2)$, $(n_1^c, n_2^c, n_3^c, n_4^c) = (4, 3, 1, 1)$.



A section in type A

Let $(a_1, \ldots, a_n) \in A_{GL_n,P} = \bigoplus_{j=1}^{n} H^0(X, \Omega^j((j - m_{j-1})x)$.
Define (E, ϕ) by:

$$E = \mathcal{O} \oplus L^{-1}((m_1 - 1)x) \oplus L^{-2}((m_2 - 1)x) \oplus \cdots \oplus L^{-(n-1)}((m_{n-1} - 1)x),$$

where $L = \Omega(x)$ and

$$\phi = \begin{bmatrix} -c_1 & -c_2 & -c_3 & \cdots & -c_n \\ t^{-\epsilon_1} & 0 & 0 & \cdots & 0 \\ 0 & t^{-\epsilon_2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & t^{-\epsilon_{n-1}} & 0 \end{bmatrix},$$

where $\epsilon_j = 1 - (m_j - m_{j-1}) = 0$ or 1, t is a global section of $\mathcal{O}(x)$ vanishing at x and $c_j = a_j t^{j - m_{j-1}} \in H^0(X, \Omega^j)$.

A section in type A

Properties:

- There is a unique reduction of E to P over x such that $Res_x(\phi) \in \mathfrak{n}$
- $Res_x(\phi)$ has Jordan blocks of sizes $n_1^c, n_2^c, \ldots, n_m^c$
- In terms of spectral data, $(E, \phi) \leftrightarrow \mathcal{O} \in Jac(\widetilde{S_a})$
- $\det(E) = \Omega^{-\frac{1}{2}n(n-1)} \otimes \left(\mathcal{O}(x)^{-\dim(G/P)}\right)$
- $\deg(E) = -\dim(G/P) (\mathrm{mod}\ n)$

If $\dim(G/P)$ is a multiple of n, can find a line bundle A on X so that $\det(E \otimes A) = \mathcal{O} \Rightarrow$ we get a section of the SL_n-Hitchin map. If $\dim(G/P)$ is not a multiple of n then we can't do this.

Conjecture: if $n \nmid \dim(G/P)$, there are no sections $s : A_{SL_n,P} \to M_{SL_n,P}$.