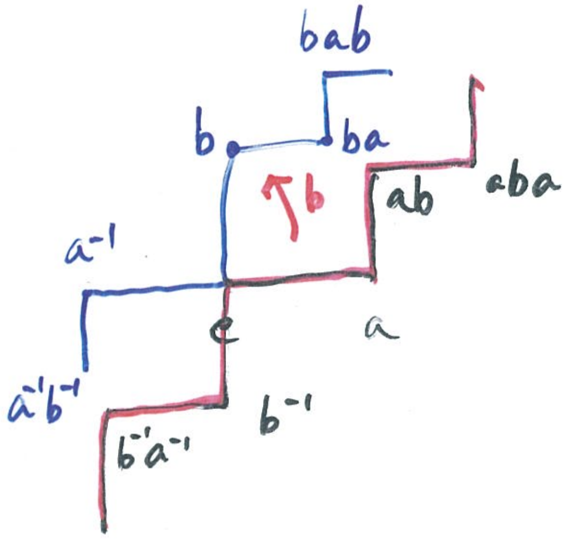
Primitive stable representations

InKang Kim

$F = F_n$ free gp with $Y = \{x_1, \cdots, x_n\}$ $n \geq 2$ generators.



$w \in F$ **primitive** if w member of a free generating set.

Example $w = ab \in F_2$ cyclically reduced

$\Rightarrow \tilde{w}$ invariant geodesic in C_F

$\tilde{w} = \ldots\ldots e(ab)(ab)$

C_F : Cayley graph of F_2

$$[w] = [ab] = [b(ab)b^{-1}] = [ba]$$

$\overline{W}$: F-inv set of bi-infinite geodesics determined by $[w]$

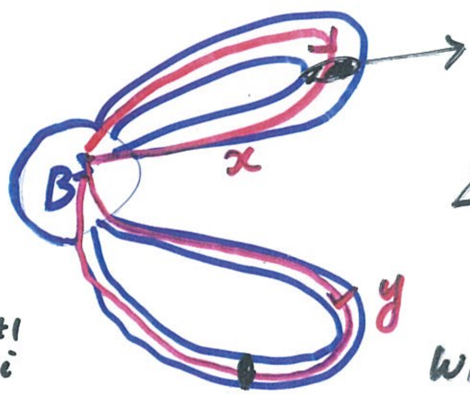
$\mathcal{P} = \{ \overline{w} \mid [w] \text{ primitive} \}$

· $wh(g, Y)$ whitehead graph of g w.r.t Y.

$g = \cdots xy \cdots$

vertices $x_i^{\pm 1}$

edges $x \to y^{-1}$



$\to$ dual disc

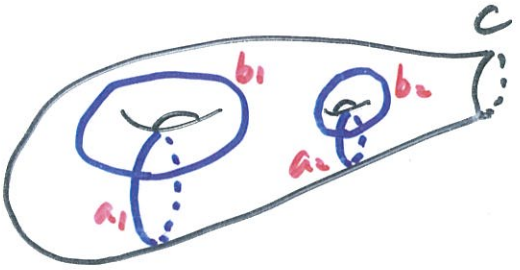
Δ = system of compression discs

vertices $\Delta_i^{\pm 1}$

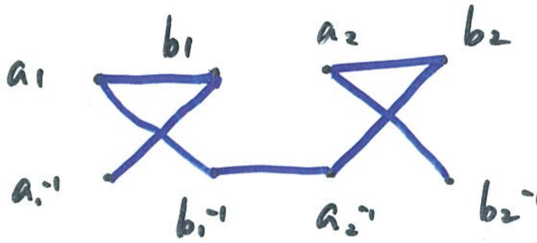
$wh(g, \Delta)$

Whitehead Lemma

$Wh(g, Y)$ connected & no cut point $\Rightarrow$ g not primitive

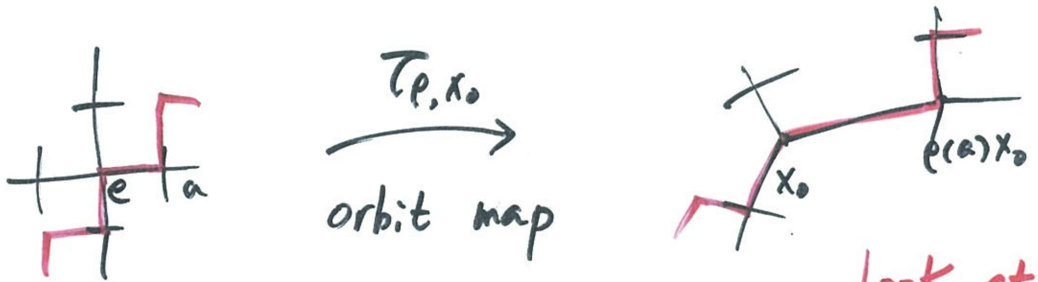


$c = \prod [a_i\ b_i] = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$



$Wh(c^2, Y)$ cycle $\Rightarrow$ c^2 not appear as a subword of any primitive element.

- $\rho : F \to G$ $\quad X = G/K$ $\quad x_0 \in X$ fixed base point



look at the images of primitive geodesics

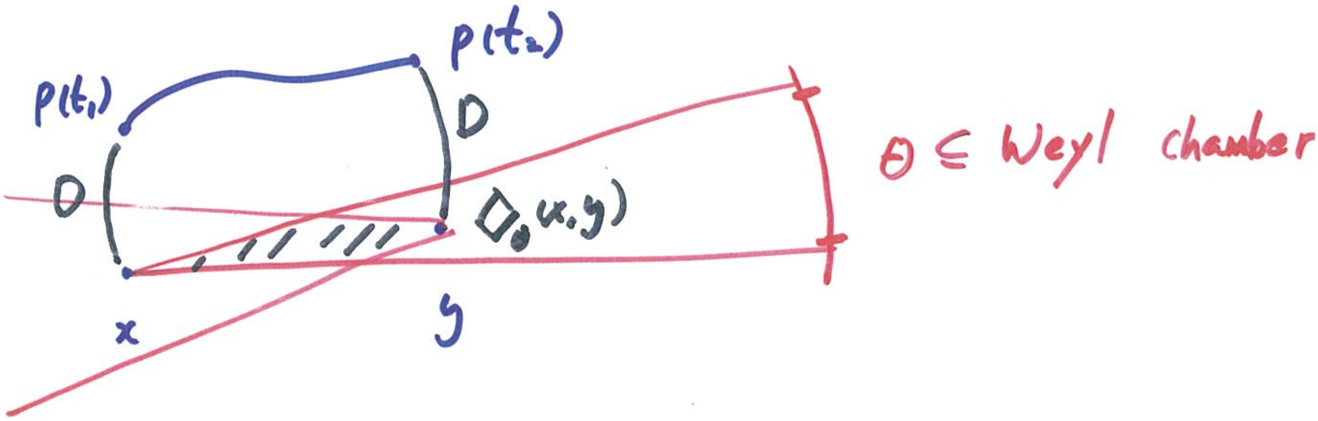
Def (Minsky) $\rho : F_n \to G$ primitive stable if $\exists (L, A, \theta, D)$ s.t $\forall$ primitive geodesic $\bar{w} \in P$

$\tau_{\rho, x_0}(\bar{w})$ is (L, A, θ, D) - Morse quasi-geodesic in X

(Kapovich - Leeb - Porti)

$p : I \to X$ $\quad (L, A)$ quasi-geodesic to control the direction ($\gamma(t) = t\begin{pmatrix}\cos(\ln t) \\ \sin(\ln t)\end{pmatrix}$ spiraling q.g in $\mathbb{R}^2$)

$\forall\, t_1, t_2 \in I$, $\quad p|_{[t_1, t_2]}$ is D-close to $\diamondsuit_\theta(x,y)$



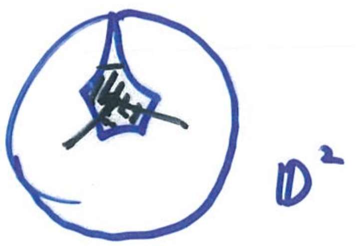
Example · convex cocpt reps in rank 1 G

$$C_\rho \xrightarrow{\tau_{\rho,x}} X \quad \text{quasi-isometric embedding}$$

· $\rho : \pi_1(\Sigma) \longrightarrow PSL_2(\mathbb{R})$ finite volume 1-cusped hyp surface

$\exists$ cpt $K \ni$ all primitive geodesics

(if γ penetrates deep into cusp γ will pick up c^2 forbidden for primitive)



C_ρ quasi-iso to $\widetilde{K}$.

Question: what about higher rank case?

Properties

① $\mathcal{PS} \circlearrowleft \mathrm{Out}(F)$
open prop. dis

② $\rho : A * B \to G$ primitive stab
$\Rightarrow \rho|_A$ convex cpt in rk 1
Anosov in rk ≥ 2

Criterion for primitive stability

Thm (Jeon - K - Lecuire - Ohshika) $\rho : F_n \to PSL_2(\mathbb{C})$
discrete, faithful, geom. inf
is primitive stable $\Leftrightarrow$ $\forall$ parabolic & ending lamination are disc-busting on ∂H, H : handlebody $\exists \eta$ s.t. $i(\lambda, \partial A) > \eta > 0$ $\forall$ ess. disc A

Non-example : $M = \Sigma \times I /$ psedo-Anos Σ : once-punctured surface

Kleinian gp corresponding to Σ
is not primitive stable.

λ ending lamination

$H = \Sigma \times I$

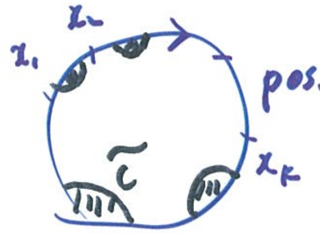
$i(\lambda, \partial(l \times I)) \to 0$

take a segment l almost parallel to λ
whose end points in $C \Rightarrow l \times I$ compression disc

Higher rank case Σ once punctured

$F = \pi_1 \Sigma$

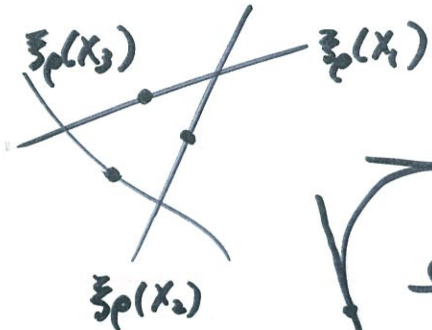
$\rho : \pi_1 \Sigma \longrightarrow G$ positive rep if $\exists$ ρ-equiv continuous positive map $\xi_\rho : \partial_\infty F \to G/B$ B: Borel

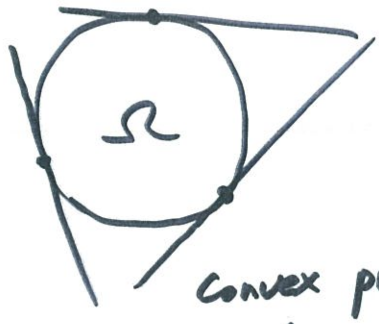


pos. oriented $(x_1 \cdots x_k) \to (\xi_\rho(x_1) \cdots \xi_\rho(x_k))$ pos conf of flags

Fock - Goncharov

(example) $G = SL(3, \mathbb{R})$





convex proj str

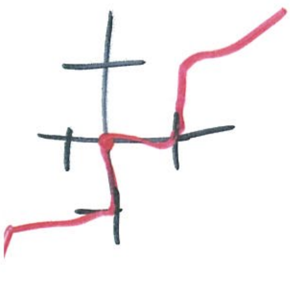
(1) ρ discrete & faithful

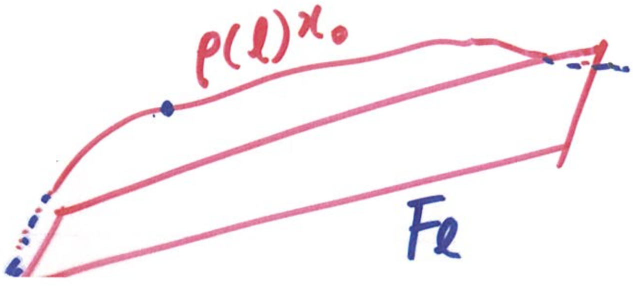
(2) $\rho(\gamma)$ positive diagonal for γ non-peripheral

(3) ξ_ρ injective except possibly $\xi_\rho(c^+) = \xi_\rho(c^-)$

Observation $\mathcal{P}_e \xrightarrow{f} \mathbb{R}^+$

$\ell \longmapsto d_X(x_0, F_\ell)$ F_ℓ : max flat determined by $\xi_\rho(\ell(\pm\infty))$





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If $d_X(x_0, F_{l_n}) \to \infty$, $l_n \to l_\infty$ in Cayley Graph

$l_\infty(\pm\infty) \notin \tilde{C}(\pm\infty)$

$\Rightarrow F_{l_n} \to F_{l_\infty}$

($\because$ otherwise l_n will contain C^2)

& $d_X(x_0, F_{l_n}) \to d_X(x_0, F_{l_\infty}) < \infty$. contradiction

$\Rightarrow \exists\ D > 0$ s.t $d_X(\rho(l)x_0, F_l) < D \quad \forall\, l \in \mathcal{P}_e$

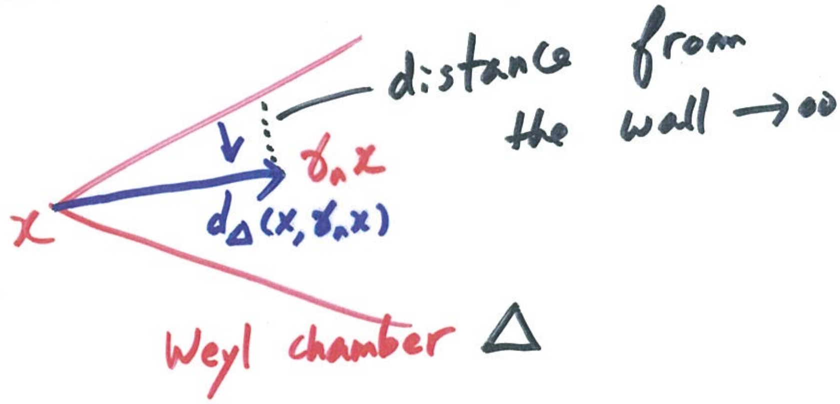
<u>observation</u> $\quad \Gamma \leq SL(3, \mathbb{R}) \qquad \overline{\Gamma} \leq \mathbb{P}(\mathrm{End}(\mathbb{R}^3))$

$(g_n) \subseteq \Gamma$ $\qquad g_n \to g_\infty \in \overline{\Gamma}$

rk 1 seq if rk $g_\infty = 1$.

Γ σ_{mod}-regular $\Leftrightarrow$ Every seq is rk 1 seq.

(γ_n) $d(d_\Delta(x, \gamma_n x), \partial\Delta) \to \infty$



σ_{mod}-reg $\Rightarrow$

distance from wall $|\lambda_1^n - \lambda_2^n| \to \infty$

$|\lambda_2^n - \lambda_3^n| \to \infty$

$$\gamma_n = k_n \begin{pmatrix} e^{\lambda_1^n} & 0 & 0 \\ 0 & e^{\lambda_2^n} & 0 \\ 0 & 0 & e^{\lambda_3^n} \end{pmatrix} k_n'$$

$k_n \to k_\infty$, $k_n' \to k_\infty'$

$Kx = x$ Cartan dec

$$\Rightarrow \begin{pmatrix} e^{\lambda_1^n - \lambda_1^n} & & \\ & e^{\lambda_2^n - \lambda_1^n} & \\ & & e^{\lambda_3^n - \lambda_1^n} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix}$$

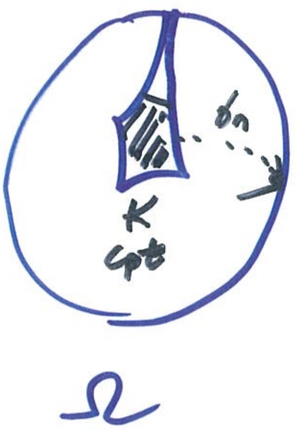
rk 1 element

Thm (K-Kim) $\rho : \pi_1 \Sigma \to PGL(n, \mathbb{R})$ positive (7)

Then ρ is σ-primitive stable.

Idea for $PGL(3, \mathbb{R}) = SL(3, \mathbb{R})$ case

ρ holonomy of strictly convex proj str w/ cusp



$\gamma_n \to \infty$

$\gamma_n K \to$ point on $\partial\Omega$ i.e.

γ_n is rk 1 sep $\Longleftrightarrow$ $\rho(\pi_1 \Sigma) \leq SL(3,\mathbb{R})$ σ-reg

$\Rightarrow$ Given $C > 0$ $\exists$ R s.t.

$$d(d_\Delta(x_0, \gamma x_0), \partial\Delta) \geq C$$

if $|\gamma| \geq R$ (action via ρ)

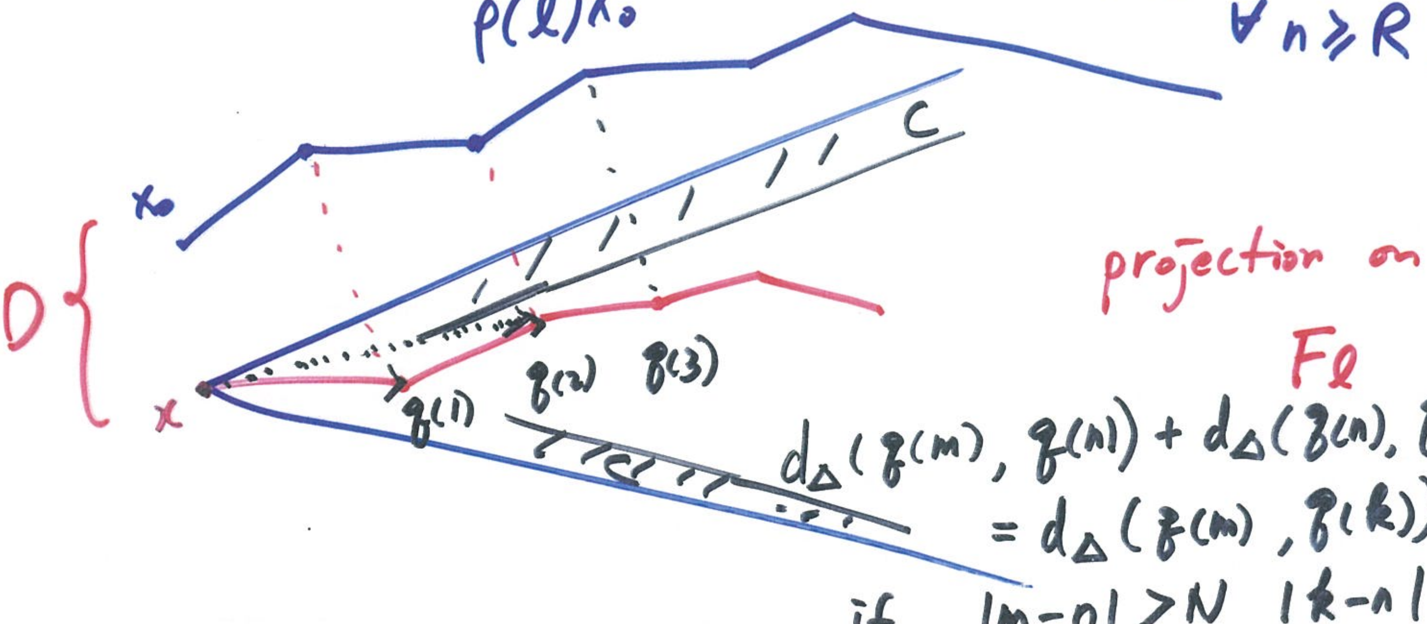
$\uparrow$ word length

$\Downarrow$

$$d_X(g(m)x, g(m+n)x) \geq C'$$

$\forall n \geq R$.

$\ell \in P_e$



$$d_\Delta(g(m), g(n)) + d_\Delta(g(n), g(k)) = d_\Delta(g(m), g(k))$$

if $|m-n| > N$, $|k-n| > N$

$\gamma|_{n_0\mathbb{Z}}$ quasi-geodesic and since $\exists$ only finitely many elemts in $\rho(\pi_1\Sigma)$ w/ word length n_0, $\exists$ cpt $\Theta \subset \mathrm{int}(\sigma_{mod})$ s.t

$\overline{\gamma(n)\gamma(n+n_0)}$ Θ reg

$\Rightarrow \gamma|_{n_0\mathbb{Z}}$ (n_0, Θ)-local Morse quasi-geodesic

$\Rightarrow \gamma$ (L, A, Θ', D') – Morse quasi-geodesic

N.B. For $GL(n,\mathbb{R}) \supseteq \Gamma$ Γ σ_{mod}-reg $\Leftrightarrow$ $\gamma_k \to \infty$ γ_k has a unique limit point in $GL(n)/B$

What about general Gromov hyp gp other than free gp?

$\Gamma = A * B$ free product decomposition

$\gamma \in$ $\underline{\gamma \text{ is separable}}$ if conjugate of γ belongs to either A or B.

<u>Example</u> compression body

$\pi_1(S_1) * \pi_1(S_2) * \pi_1(S_3)$

$\rho : \pi_1(\text{compression body}) \to G$

separable stable if all the sepable geodesic are mapped to uniform Morse quasi-geodesics by orbit map.

(M. Lee)

$SS \circlearrowleft Out(\pi_1 M)$ prop. dis

Thm (K-Porti) F free gp

$$\mathcal{P}S = SS$$

Def: $\rho : \Gamma \to G$ uniformly separable stable if for some $\langle S \rangle = \Gamma$ & $x_0 \in G/K = X$ $\forall \Phi \in Aut(\Gamma)$ for any subgp F_i generated by a subset of $\Phi(S)$ which admits free decomposition, $\exists$ uniform $(L\ A\ \Theta\ D)$ s.t. $\rho|_{F_i} : F_i \to G$ is separable stable with uniform constant $(L\ A\ \Theta\ D)$. $USS \circlearrowleft Out(\Gamma)$ p.d

<u>Example</u> S closed hyp surface $\Gamma = \pi_1(S)$

c

$\pi_1(S \setminus c)$ free gp

S $g \geq 3$ closed

Thm. $\mathcal{UPS}(SL(2,\mathbb{C})) \cap AH(S, SL(2,\mathbb{C}))$
$= $ quasi-fuchsian space QF

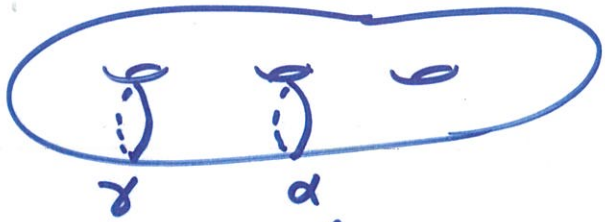
(Pf) $\rho \in \partial QF$ uniformly primitive stable

$\Rightarrow N(\rho)$ neighborhood $\subseteq \mathcal{UPS}$

cusps are dense $\Rightarrow \exists \underset{ups}{\rho'}$ s.t. $\rho'(\gamma)$ parabolic

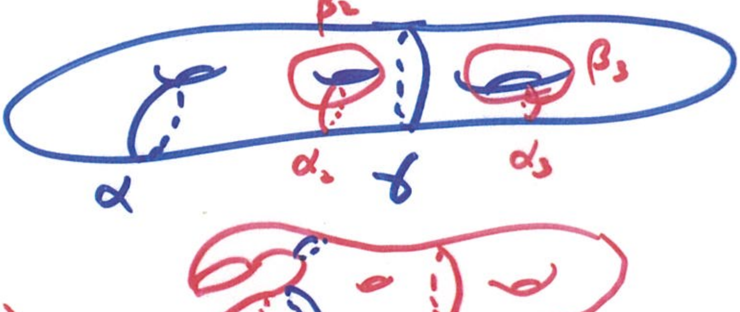
(A) γ non-separating

$\underset{\text{primitive}}{\gamma} \in \pi_1(S \setminus \alpha)$
$=$ free gp



$\rho'|_{\pi_1(S\setminus\alpha)}$ primitive stable $\Rightarrow \rho'(\gamma)$ cannot be parabolic

(B) γ separating



$\gamma \in \pi_1(S \setminus \alpha)$
$=$
$A * B$
$\gamma \in$ $=$
$\langle \alpha_2\, \beta_2\, \alpha_3\, \beta_3 \rangle * \langle \alpha \rangle$

$= A * B$

$\rho'|_{\pi_1(S\setminus\alpha)} : \pi_1(S\setminus\alpha) \longrightarrow SL(2,\mathbb{C})$ primitive stable

$\Rightarrow \rho'|_A$ convex cpt

$\Rightarrow \rho'(\gamma)$ cannot be parabolic.

Conj: $\mathcal{UPS}(SL(2,\mathbb{C}))$
$= QF$