Anti-de Sitter geometry and polyhedra inscribed in quadrics

Joint work with Jeff Danciger and Sara Maloni

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Workshop on moduli spaces of geometric structures
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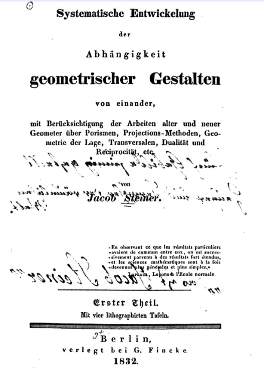
Modern questions on polyhedra



Systematische Entwickelung
der
Abhängigkeit
geometrischer Gestalten
von einander,

mit Berücksichtigung der Arbeiten alter und neuer Geometer über Porismen, Projections-Methoden, Geometrie der Lage, Transversalen, Dualität und Reciprocität, etc.

von
Jacob Steiner.

«En observant ce que les résultats particuliers avaient de commun entre eux, on est successivement parvenu à des résultats fort étendus, et les sciences mathématiques sont à la fois devenues plus générales et plus simples.»
Laplace, Leçons à l'École normale.

Erster Theil.
Mit vier lithographirten Tafeln.

Berlin,
verlegt bei G. Fincke.
1832.

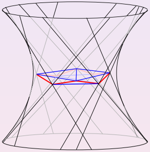
Jakob Steiner (1796–1863) started the study of the combinatorics of Euclidean/projective polyhedra. He introduced new questions (1832).

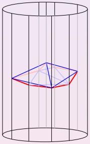
- *What are the possible* combinatorics *of polyhedra ?*

Ernst Steinitz (1871–1928), 1916 : *the planar 3-connected graphs.*

Inscribed polyhedra

77) Wenn irgend ein convexes Polyëder gegeben ist, läſst sich dann immer (oder in welchen Fällen nur) irgend ein anderes, welches mit ihm in Hinsicht der Art und der Zusammensetzung der Grenzflächen übereinstimmt (oder von gleicher Gattung ist), in oder um eine Kugelfläche, oder in oder um irgend eine andere Fläche zweiten Grades beschreiben (d. h. daſs seine Ecken alle in dieser Fläche liegen, oder seine Grenzflächen alle diese Fläche berühren)?





- *What are the possible combinatorics of polyhedra inscribed in the sphere, or in another quadric ?*

The question is *projectively invariant* so there are only 3 quadrics to consider : the sphere, one-sheeted hyperboloid, and cylinder.

Steinitz (1927) : *there are polyhedra not inscribable in a sphere.*

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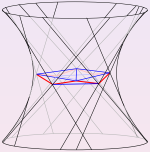
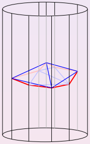
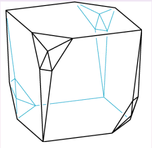
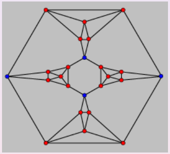

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In 1993, Craig Hodgson, Igor Rivin and Warren Smith gave a partial answer to Steiner's question.

A polyhedron can be inscribed in a sphere iff a certain system of linear equations and inequalities has a solution.

Consequence : deciding whether a graph can be realized can be decided in polynomial time.

Their result is based ideal hyperbolic polyhedra.

An answer to Steiner's question

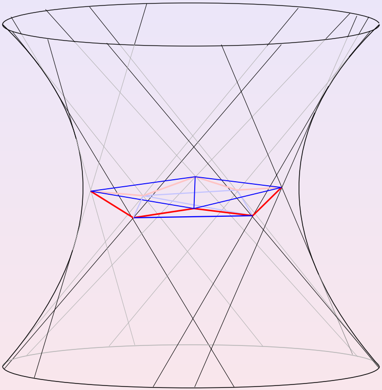

Theorem A (Danciger-Maloni-S.)

A planar 3-connected graph can be realized as a polyhedron inscribed in the hyperboloid (resp. cylinder) iff it is inscribable in a sphere and it admits a Hamiltonian cycle.

A *Hamiltonian cycle* is a path going through each vertex exactly once.
Note : harder to decide whether a polyhedron is inscribable in a hyperboloid or in a cylinder than in a sphere.

Ideal polyhedra in AdS_3

Ideal polyhedra have vertices on $\partial_\infty AdS_3$ and edges in AdS_3. So their faces are space-like (hyperbolic).
Two types of faces : past and future. "Equator" is a Hamiltonian cycle, with angles $w < 0$.



Main statements

Theorem A (Danciger-Maloni-S.)

A planar 3-connected graph can be realized as a polyhedron inscribed in the hyperboloid (resp. cylinder) iff it is inscribable in a sphere and it admits a Hamiltonian cycle.

Theorem B (Andreev, Rivin)

Let $\Gamma \subset S^2$ be the 1-skeleton of a cell decomposition, and let $w : \Gamma_1 \to (0, \pi)$. $\exists P \subset H^3$ with exterior dihedral angles w iff

- *i.* $\forall v \in \Gamma_0, \sum_{e \in v} w(e) = 2\pi$,
- *ii.* $\forall$ *other path* c *in* Γ^*, $\sum_{e \in c} w(e) > 2\pi$.

Theorem C (Danciger, Maloni, S.)

Let $\Gamma \subset S^2$ be the 1-skeleton of a cell decomposition, and let $w : \Gamma_1 \to \mathbb{R}_{\neq 0}$. $\exists P \subset AdS_3$ with exterior dihedral angles w iff

- *i.* $w < 0$ *on a Hamiltonian path* $\gamma \subset \Gamma_1$,
- *ii.* $\forall v \in \Gamma_0, \sum_{e \in v} w(e) = 0$,
- *iii.* $\forall$ *other path* c *in* Γ^* *intersecting* γ *twice*, $\sum_{e \in c} w(e) > 0$.