Geometric realisation for degenerations of hyperbolic structures

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Background and Motivation (the case of dimension 2)

An interpretation of the Thurston compactification of Teichmüller space using **R**-trees

Recall that the Thurston compactification of Teichmüller space $T(S)$ is defined as follows:

For each $(R, f) \in T(S)$, regard R as a hyperbolic surface putting a hyperbolic metric compatible with the conformal structure of R.

Let $\mathcal{S}$ be the set of isotopy classes of non-contractible simple closed curves on S.

Define a map $I\colon T(S) \to \mathbb{R}_+^{\mathcal{S}}$ by setting the s-coordinate of $I(R, f)$ to be the length of closed geodesic on R homotopic to $f(s)$.
Let $P\mathbb{R}_+^{\mathcal{S}}$ be the projectivisation of $\mathbb{R}_+^{\mathcal{S}}$, and let π be the projection.

We also embed the measured foliation space $\mathcal{MF}(S)$ (or equivalently the measured lamination space $\mathcal{ML}(S)$) into $\mathbb{R}_+^{\mathcal{S}}$ by defining $I_M(F, \mu)$ to be $\inf_{\sigma \in s} \int_\sigma d\mu$.

Theorem (Thurston 1978). The map $\pi \circ I$ is injective and its image is relatively compact. Its closure in $P\mathbb{R}_+^{\mathcal{S}}$ is homeomorphic to $\mathbb{B}^{6g-6}$, and its boundary coincides with the image of $\pi \circ I_M$.

Morgan-Shalen (1984) gave a reinterpretation of this compactification using valuations and isometric actions of π_1(S) on **R**-trees.
Bestivna and Paulin (1988) showed that **R**-tree actions can be interpreted as equivariant Gromov limits, bypassing valuations.

Let $\{m_i\} \in \mathcal{T}(S)$ be a divergent sequence. Regard each m_i as a Fuchsian representation $\phi_i \colon \pi_1(S) \to \mathrm{PSL}_2\,\mathbb{R}$.

Fix a generator system $\{\gamma_1, \ldots, \gamma_n\}$ of $\pi_1(S)$, and a basepoint x_0 in $\mathbb{H}^2$. We conjugate ϕ_i so that the function $\max_{k=1}^n d(x, \phi_i(\gamma_k)x)$ defined for $x \in \mathbb{H}^2$ takes a minimum value at x_0.

Let L_i be $\max_i d(x_0, \phi_i(\gamma_i)x_0)$. Since $\{\phi_i\}$ is divergent, we have $L_i \to \infty$. We consider the rescaled hyperbolic space $\frac{1}{L_i}\mathbb{H}^2$, keeping the action of ϕ_i the same. Then it can be proved that the action of $\pi_1(S)$ on $\frac{1}{L_i}\mathbb{H}^2$ converges to an isometric action of $\pi_1(S)$ on an $\mathbb{R}$-tree.

An isometric action on an **R**-tree is said to have small edge-stabilisers, when the stabiliser of every non-trivial arc is virtually abelian.
Using the discreteness criterion of Shimizu-Jorgensen, we can see that any limit **R**-tree action for divergent Fuchsian representations has small edge-stabilisers.

Skora's theorem (1996)

Theorem(Skora) Suppose that S is a closed orientable surface of genus at least 2. Let T be an $\mathbf{R}$-tree on which $\pi_1(S)$ acts by isometries with small edge-stabilisers. Then, there are a measured lamination λ on S and an isometry from the dual $\mathbb{R}$-tree T_λ to T which is equivariant with respect to the standard action of $\pi_1(S)$ on T_λ and the given action of $\pi_1(S)$ on T.

An $\mathbb{R}$-tree T_λ is said to be dual to a measured lamination λ (or its equivalent measured foliation) when it is isometric to the leaf space of the lift of the measured foliation equivalent to λ to the universal cover $\mathbb{H}^2$ of S. The covering translation induces the isometric action of $\pi_1(S)$ on T_λ.

Combining the theory of Morgan-Shalen-Bestvina-Paulin with Skora's theorem, we get an alternative interpretation of the Thurston compactification as follows.

For a divergent sequence $\{m_i \in \mathcal{T}(S)\}$, consider its limit $\mathbb{R}$-tree T on which $\pi_1(S)$ acts by isometries. By Skora's theorem, there is a measured lamination (or foliation) λ such that T is $\pi_1(S)$-equivariantly isometric to the dual tree T_λ. The projective class of λ is regarded as a point on the boundary to which $\{m_i\}$ converges. In fact, by expressing the translation lengths on T_λ by the intersection number with λ, this correspondence is justified.

Note that the theory of Morgan-Shalen-Bestvina-Paulin works even for dimension 3: i.e. for degeneration of hyperbolic 3-manifolds, or even for higher dimensions.

So we can pose the following natural questions:

1. Can we realise degeneration of 3-dimensional hyperbolic structures (on the interior of a compact 3-manifold with incompressible boundary) by a measured lamination (a codimension-1 incompressible measured lamination in a 3-manifold) ?

2. In particular, can we generalise Skora's theorem to dimension 3?

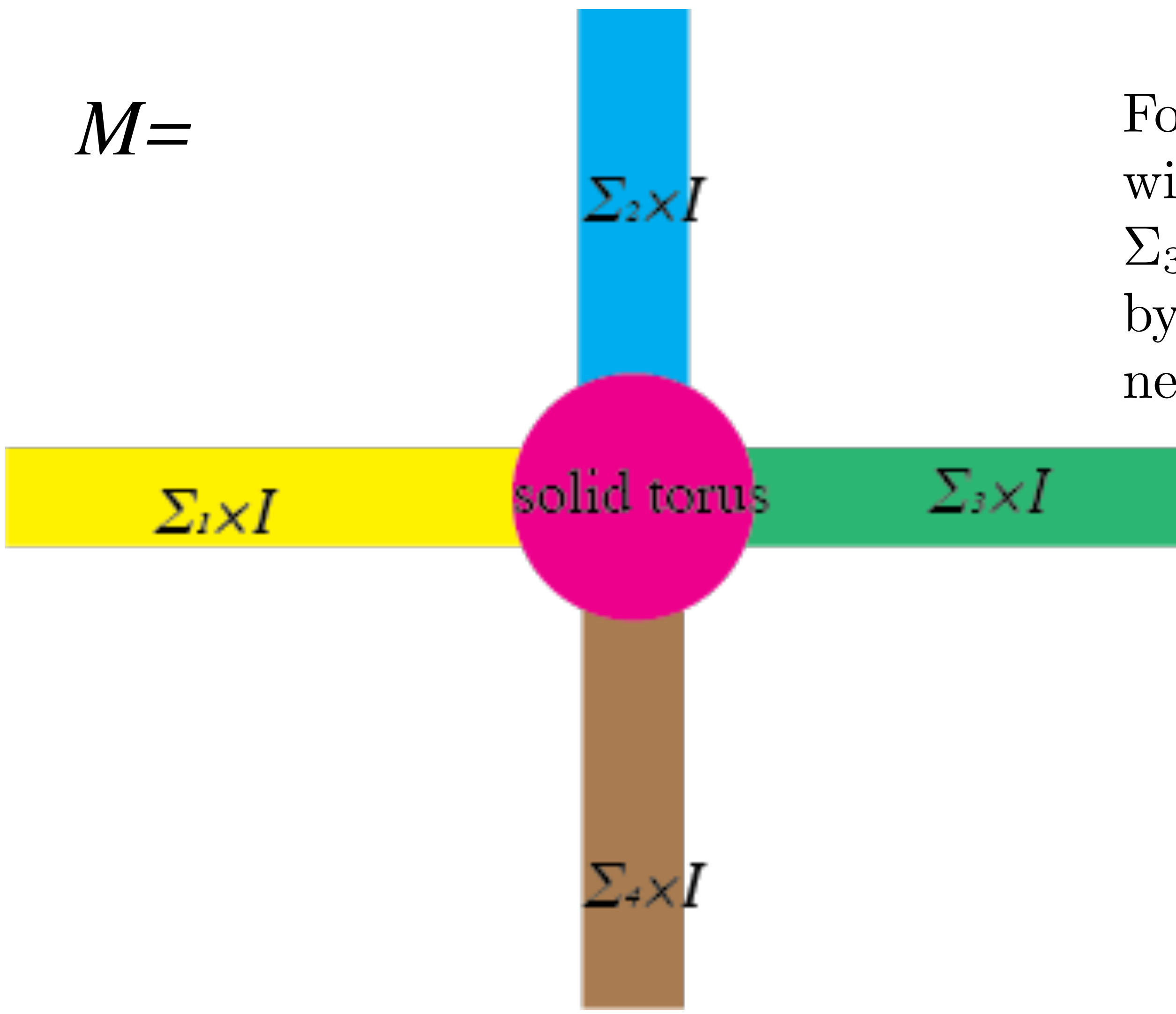
To put it more precisely:

1. Suppose that there is a sequence of divergent hyperbolic structures (m_i) on the interior of a compact 3-manfold M with incompressible boundary. By the theory of Bestvina-Paulin, passing to a subsequence, there is a rescaled Gromov limit of (m_i), which is an $\mathbf{R}$-tree T on which $\pi_1(M)$ acts. Then, is there an incompressible lamination properly embedded in M whose dual tree is isometric to T equivariantly with respect to the actions of $\pi_1(M)$?
2. More in general, if $\pi_1(M)$ acts on an $\mathbf{R}$-tree T with small edge-stablisers, is there an incompressible lamination properly embedded in M whose dual tree is isometric to T equivariantly with respect to the actions of $\pi_1(M)$?

The answer is ‘no’ to both, in general.

A counterexample: a book of I-bundles (with more than three pages) (named by Anderson-Canary: They used this manifold to show the existence of bumping in deformation spaces.)

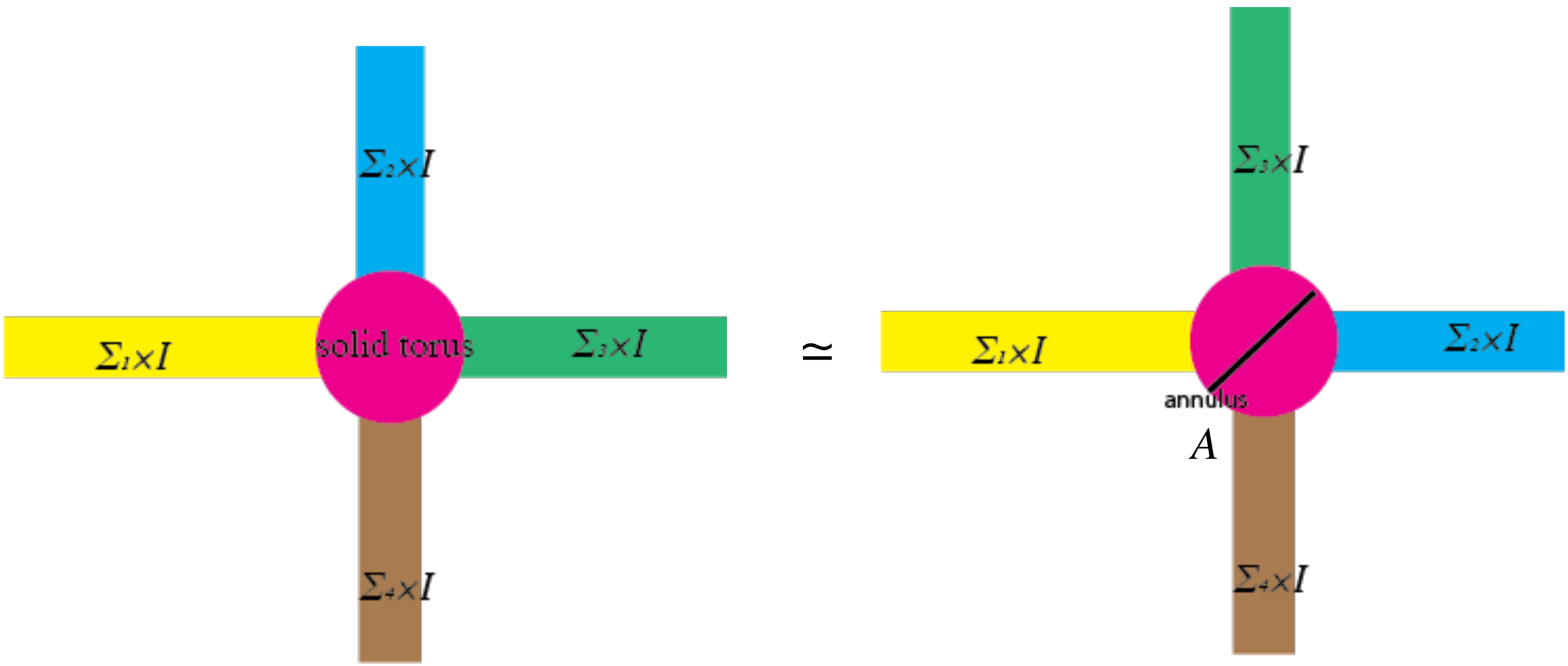
$M=$



Four product I-bundles over distinct surfaces with one boundary component $\Sigma_1 \times I, \Sigma_2 \times I$, $\Sigma_3 \times I$ and $\Sigma_4 \times I$ are attached to a solid torus by pasting the $\partial \Sigma_j \times I$ to disjoint regular neighbourhoods of longitudes.

M is homotopy equivalent to M' which is obtained by rearranging pages.

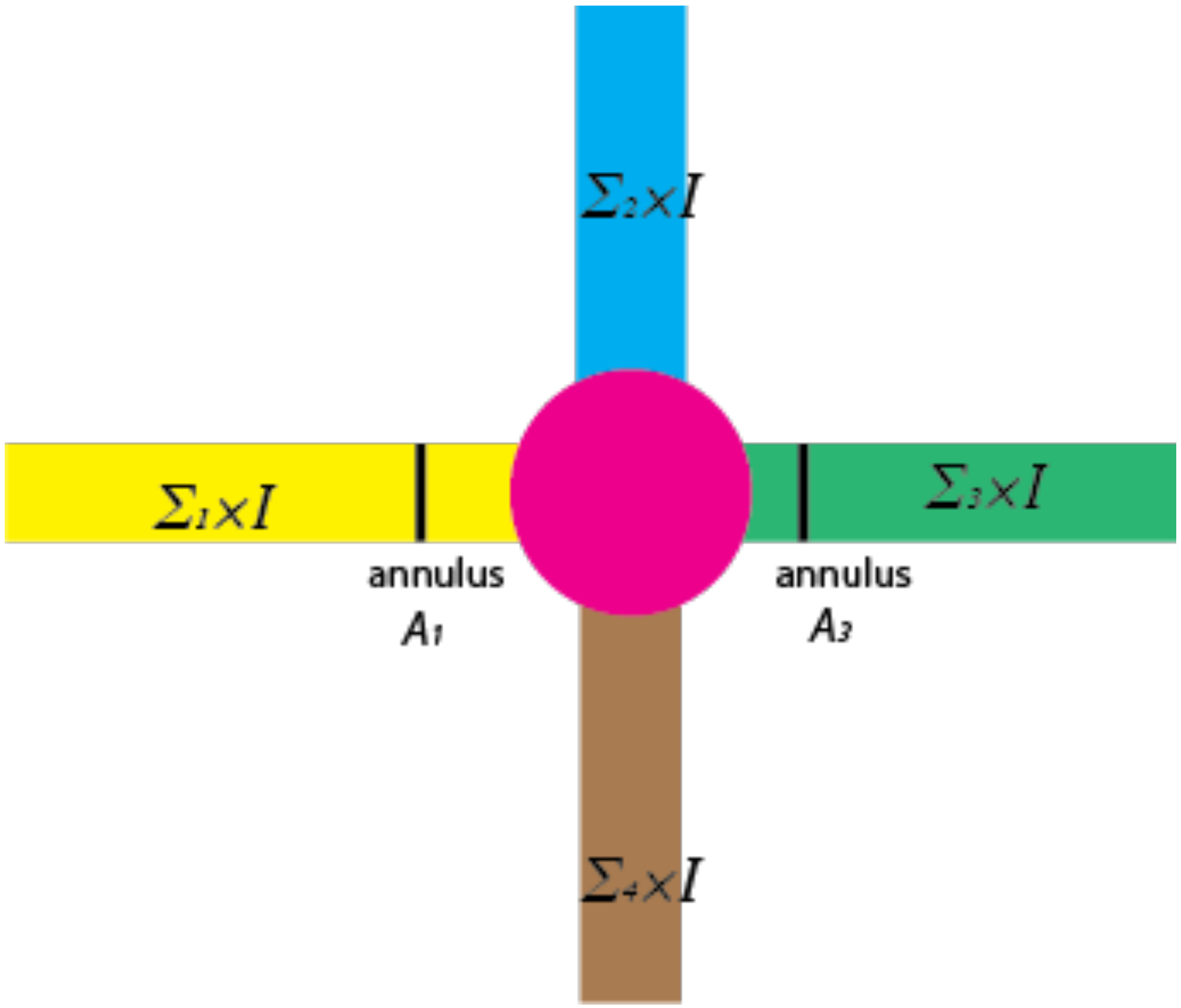
Put an incompressible annulus A into M' separating $\Sigma_1 \times I$, $\Sigma_3 \times I$ from $\Sigma_2 \times I$, $\Sigma_4 \times I$.



A tree dual to this annulus on which $\pi_1(M') \cong \pi_1(M)$ acts by isometries is not dual to any incompressible lamination in M.

The tree dual to the annulus in M' can appear as a (rescaled) Gromov limit of hyperbolic structures on Int M.

Start from a convex cocompact hyperbolic structure on Int M.



Put two incompressible annuli, A_1 in $\Sigma_1 \times I$, and A_3 in $\Sigma_3 \times I$.

Deform the hyperbolic structure by performing the n-times iterated Dehn twist along A_1 and the $(-n)$-times iterated Dehn twist along A_3 to get a sequence (m_n) in the deformation space.

The sequence (m_n) diverges in the deformation space, and its rescaled Groom limit is dual to the annulus A in M'.

Review on Jaco-Shalen-Johannson theory

Every Haken manifold M with incompressible boundary has a disjoint collection of Seifert pairs, I-pairs, and solid torus pair, which is unique up to isotopy and has the following properties.

A Seifert pair is a Seifert fired manifold $N \subset M$ whose frontier consists of fibred tori which are incompressible surface in M.

An I-pair is an I-bundle N over a compact surface such that $N \cap \partial M$ is its associated ∂I-bundle.

A solid torus pair V is embedded in M in such a way that $V \cap \partial M$ consists of parallel essential annuli on ∂V and $\mathrm{Cl}(\partial V \setminus \partial M)$ consists of essential annuli in M.

Every 3-manifold homotopy equivalent to M is obtained by repeating flipping on I-pairs and shuffling on solid torus pairs.

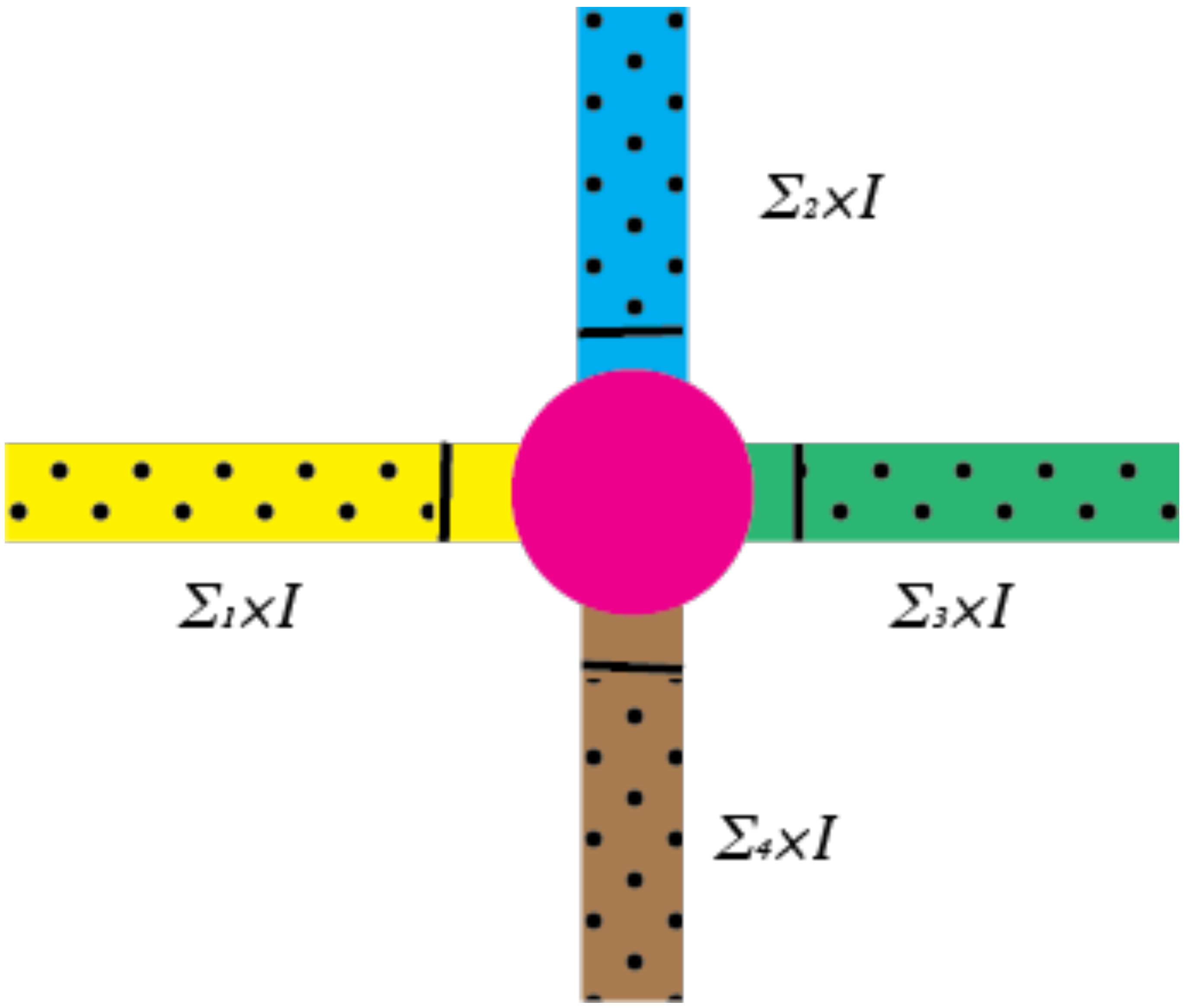
Thurston's 'broken windows only' theorem

Suppose that (M,P) is a 'pared manifold' with incompressible boundary: i.e. P consists of incompressible tori and annuli lying on the boundary of M, and every incompressible map from a torus to M or from an annulus to (M, P) is homotoped into P.

A window is a union of I-pairs and regular neighbourhoods of the frontiers of solid torus pairs which cannot be homotoped into other I-pairs. A window is an I-bundle over a surface, which is denoted by $\mathrm{wb}(M,P)$.

Theorem (Thurston 1986). If $\Gamma \subset \pi_1(M)$ is any subgroup which is conjugate to the fundamental group of a component of M − window(M, P), then the set of representations of Γ in $\mathrm{Isom}(\mathbf{H}^3)$ induced from $\mathrm{AH}(M,P)$ are bounded, up to conjugacy.
Given any sequence $N_i \in \mathrm{AH}(M,P)$, there is a subsurface with incompressible boundary $x \subset \mathrm{wb}(M,P)$ and a subsequence $N_{i(j)}$ such that the restriction of the associated sequence of representations $\varrho_{i(j)} : \pi_1(M) \to \mathrm{Isom}(\mathbf{H}^3)$ to a subgroup $\Gamma \subset \pi_1(M)$ converges if and only if Γ is conjugate to the fundamental group of a component of $M - X$, where X is the total space of the interval bundle above x. Furthermore, no subsequence of $\varrho_{i(j)}$ converges on any larger subgroup.

Cf. our counterexample with the second sentence of the theorem.



Spotted parts are windows. X must be a union of some of annuli denoted by bold lines.

Since m_n restricted $\Sigma_1 \times I \cup \Sigma_3 \times I$ converges, X can contain neither of the two vertical annuli. In the same way, since m_n restricted $\Sigma_2 \times I \cup \Sigma_4 \times I$ converges, X can contain neither of two horizontal annuli. Thus X must be empty….. contradiction.

Main Theorem. Let M be a Haken manifold with incompressible boundary. Suppose that $\pi_1(M)$ acts on an $\mathbf{R}$-tree T with small edge-stabilisers. Then there are a Haken manifold M' homotopy equivalent to M which is obtained from M by shufflings around solid torus pairs, and a codimension-1 incompressible lamination L in M' approximated by weighted unions of incompressible tori and annuli such that the dual tree of L is isometric to T equivariantly with respect to the action of $\pi_1(M) \cong \pi_1(M')$.

(We call such a lamination abelian.)

Corollary. Degeneration of hyperbolic structures on the interior of a Haken 3-manifold with incompressible boundary is realised by an abelian codimension-1 incompressible lamination in a Haken manifold M' homotopy equivalent to M obtained by shufflings from M.

Outline of proof

The results of Morgan-Shalen (1988) shows that there are an abelian codimension-1 incompressible lamination L in M and an equivariant morphism f from the dual tree T_L to T.

We can remove folds at edge points easily, just by removing some redundant leaves of L.

We can show by an argument similar to Skora's that if there is a fold at vertex point, two lifts of non-compact leaves contained in I-pairs cannot be identified by f.

Two lifts of compact leaves may be identified. The assumption of small stabilisers implies that this can happen only these correspond to two annuli whose core curves are freely homotopic.
This is the situation where they lie on the opposite sides of a solid torus pair.
By changing M to a homotopy equivalent M', we can remove such folds.

Thank you very much.