Geometry, Topology and Dynamics of Moduli Spaces

# An Explicit Geometry of Moduli Spaces of Higgs Bundles and Singular Connections on a Smooth Curve and Differential Equations of Painlevé type

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14:30-15:30, August 12, 2016 Institute for Mathematical Sciences, National University of Singapore Supported by JSPS Grand-in-Aid for Scientific Research (S)24224001 1. Moduli spaces of stable  $\lambda\text{-parabolic connectios}$  1.1. Settings.

- $\bullet \ C$ : a nonsingular projective curve of genus  $g \geq 0$
- $\mathbf{t} = \{t_1, \ldots, t_n\}$ , a set of *n*-distinct points on *C*.

$$D(\mathbf{t}) = \sum_{i=1}^{n} t_i = t_1 + \dots + t_n.$$

•  $M_{g,n} = \{(C, \mathbf{t}) \text{ as above}\}/\simeq$ : The moduli of (ordered) *n*-pointed curves of genus *g*.



1.2.  $\lambda$ -connections. Fix  $\lambda \in \mathbf{C}$ .

**Definition 1.1.**  $(E, \nabla)$  is called a  $\lambda$ -connection if

- E : An algebraic vector bundle on C of rank r and of degree d.
- $\nabla : E \longrightarrow E \otimes \Omega^1_C(D(\mathbf{t}))$ : A logarithmic  $\lambda$ -connection.  $a \in \mathcal{O}_C, \sigma \in E$

$$\nabla(a\sigma) = \lambda\sigma \otimes da + a\nabla(\sigma) \quad \text{$\lambda$-twisted Leibniz rule}$$

We denote by

$$L = \Omega^1_C(D(\mathbf{t}))$$

the line bundle or the invertible sheaf of meromorphic 1 form on C having poles on  $D(\mathbf{t}) = t_1 + t_2 + \cdots + t_n$  at most order 1. Later we may allow the higher order pole  $D(\mathbf{t}) = m_1 t_1 + m_2 t_2 + \cdots + m_n t_n$  with  $m_i \ge 1$ .  $\deg L = 2g - 2 + n$ . We assume that  $n \ge 1$  by a technical reason.

• 
$$\lambda \neq 0$$
: linear connection:  
 $(E, \nabla)$ :  $\lambda$ -connection  $\Rightarrow (E, \frac{1}{\lambda}\nabla)$ : a usual connection  
Locally near at  $z = t_i$ , taking a local frame of  $E$  near  $z = t_i$ ,  
 $E \simeq \mathcal{O}_{C,t_i}^{\oplus r} \ni (a_k(z))_{k=1}^r$ ,  $A(z)\frac{dz}{z-t_i} \in M_r(\mathcal{O}_{C,t_i}) \otimes \Omega_C^1(D(\mathbf{t}))$   
 $\nabla((a_k(z))) = \lambda(da_k(z)) + A(z)(a_k(z))\frac{dz}{z-t_i}$ 

• 
$$\lambda = 0$$
: Higgs bundle: Denote  $\nabla = \Phi$ .  
 $(E, \Phi)$ : 0-connection  $\Rightarrow (E, \Phi)$  :a Higgs bundle,  $\Phi$ :Higgs field  
Twisted Leibniz rule leads: for a local section  $a \in \mathcal{O}_C, \sigma \in E$ 

$$\Phi(a\sigma) = a\Phi(\sigma) \quad \text{an } \mathcal{O}_C\text{-linear hom.}$$

 $\Phi \in \operatorname{End}(E) \otimes L$ . Locally near  $z = t_i$ ,  $B(z) \frac{dz}{z - t_i} \in \operatorname{M}_r(\mathcal{O}_{C, t_i}) \otimes L$ .

$$\Phi((a_k(z))) = B(z)(a_k(z))\frac{dz}{z-t_i}$$

- $1.3. \ \mbox{Residues}$  and Local exponets.
  - $\bullet \; (E, \nabla) \text{, } (E, \Phi)$  as above.
  - $\operatorname{res}_{t_i}(\nabla) = A(t_i), \operatorname{res}_{t_i}(\Phi) = B(t_i) \in \operatorname{End}(E_{|t_i})$ : residue homomorphisms.  $A(t_i) = (a_{kl})_{1 \leq k, l \leq r}, B(t_i) = (b_{kl})_{1 \leq k, l \leq r}$ : complex  $r \times r$  matrices.
  - $\bullet$  We put an order of eigenvalues of  ${\rm res}_{t_i}(\nabla)$  and  ${\rm res}_{t_i}(\Phi)$  respectively, and denote them as

$$\{\nu_0^{(i)}, \nu_1^{(i)}, \cdots, \nu_{r-1}^{(i)}\}$$

local exponents of  $\nabla$  at  $t_i$ .

 $\bullet$  We denote the local exponents of  $\nabla$  and  $\Phi$  by

$$\boldsymbol{\nu} = (\nu_j^{(i)})_{0 \le j \le r-1}^{1 \le i \le n}$$

1.4. Fuchs relation.

Lemma 1.1. For a  $\lambda$ -connection  $(E, \nabla)$  (resp. a Higgs bundle  $(E, \Phi)$ ), with singularity at  $D(\mathbf{t})$  as above, we have the following relation.

$$\sum_{i=1}^{n} (\sum_{j=0}^{r-1} \nu_j^{(i)}) = -\lambda \deg E = -\lambda d$$
$$\left( \operatorname{resp.} \sum_{i=1}^{n} (\sum_{j=0}^{r-1} \nu_j^{(i)}) = 0 \right)$$

1.5. The space of local exponents of  $\lambda$ -connections.  $\mathcal{N}_{r,\lambda}^{n}(d) := \left\{ \boldsymbol{\nu} = (\nu_{j}^{(i)})_{0 \leq j \leq r-1}^{1 \leq i \leq n} \in \mathbf{C}^{nr} \middle| \begin{array}{l} \lambda d + \sum_{1 \leq i \leq n} \sum_{0 \leq j \leq r-1} \nu_{j}^{(i)} = 0 \end{array} \right\}.$   $\mathcal{N}_{r,H}^{n} = \mathcal{N}_{r}^{n}(0) \quad \text{Higgs bundle case}$ 

## 1.6. Genericity for local exponents.

Definition 1.2. Let *ν* = {*ν*<sub>j</sub><sup>(i)</sup>}<sup>0≤j≤r-1</sup><sub>1≤i≤n</sub> ∈ *N*<sub>r,λ</sub><sup>n</sup>(d).
(1) *ν* is called *resonant*, if for some *i* and *j*<sub>1</sub> ≠ *j*<sub>2</sub>, *ν*<sub>j1</sub><sup>(i)</sup> − *ν*<sub>j2</sub><sup>(i)</sup> ∈ λ**Z**.
(2) *ν* is called *reducible* if there exists a subset *ν'* = {*ν*<sub>j'</sub><sup>(i)</sup>} of *ν* such that for each *i*, 1 ≤ *i* ≤ *n*, the number of *ν*<sub>j'</sub><sup>(i)</sup> ∈ *ν'* is a fixed number *k*, 1 ≤ *k* ≤ *r* − 1 and ∑<sub>*ν'*</sub>*ν*<sub>j'</sub><sup>(i)</sup> ∈ λ**Z** where the last sum is taken over *ν'*. If *ν* is not reducible, *ν* is called *irreducible*(3) If *ν* is neither resonant, nor reducible, we call *ν* is *generic*.

Remark 1.1. If a  $\lambda$ -connection  $(E, \nabla)$  has a subconnection  $(F, \nabla_{|F})$  is with  $0 < \operatorname{rank} F < \operatorname{rank} E$ , the local exponents of  $(E, \nabla)$  is reducible.

1.7. Parabolic connections.

**Definition 1.3.** Fix  $(C, \mathbf{t}) \in M_{g,n}$  and  $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)$ 

•  $(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n})$ : a  $\nu$ -parabolic connection of rank r and degree d on C

•  $(E, \nabla)$ : a logarithmic connection of rank r and degree d  $\nabla : E \longrightarrow E \otimes \Omega^1_C(D(\mathbf{t}))$ •  $l^{(i)}_* : E_{|t_i} = l^{(i)}_0 \supset l^{(i)}_1 \supset \cdots \supset l^{(i)}_{r-1} \supset l^{(i)}_r = 0$ : a filtration of  $E_{|t_i}$  for each  $i, 1 \leq i \leq n$  such that (1)  $\dim(l^{(i)}_j/l^{(i)}_{j+1}) = 1$  and (2)  $(\operatorname{res}_{t_i}(\nabla) - \nu^{(i)}_j)(l^{(i)}_j) \subset l^{(i)}_{j+1}$  for  $j = 0, 1, \cdots, r-1$ . 1.8. Parabolic stability. Next, we define  $\alpha$ -stability condition on the  $\nu$ -parabolic connections  $(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n})$ .

• Fix a sequence of rational numbers  $\alpha = (\alpha_j^{(i)})_{1 \le j \le r}^{1 \le i \le n}$  such that

(1) 
$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \dots < \alpha_r^{(i)} < 1$$
  
for  $i = 1, \dots, n$  and  $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$  for  $(i, j) \neq (i', j')$ .

• 
$$(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n})$$
: a  $\nu$ -parabolic connection.

•  $0 \subsetneqq F \subset E$ ,  $\nabla(F) \subset F \otimes \Omega^1_C(D(\mathbf{t}))$ . Define integers  $\operatorname{length}(F)^{(i)}_j$  by

(2) 
$$\operatorname{length}(F)_{j}^{(i)} = \dim(F|_{t_{i}} \cap l_{j-1}^{(i)})/(F|_{t_{i}} \cap l_{j}^{(i)}).$$

Note that  $\text{length}(E)_{j}^{(i)} = \dim(l_{j-1}^{(i)}/l_{j}^{(i)}) = 1$  for  $1 \le j \le r$ .

**Definition 1.4.** • A  $\nu$ -parabolic connection  $(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n})$ : is  $\alpha$ -stable

$$\begin{array}{c} \longleftrightarrow \\ 0 & \subsetneqq F & \subsetneqq E, \ \nabla(F) \subset F \otimes \Omega^{1}_{C}(D(\mathbf{t})), \\ \\ \frac{\deg F + \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_{j}^{(i)} \operatorname{length}(F)_{j}^{(i)}}{\operatorname{rank} F} < \frac{\deg E + \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_{j}^{(i)} \operatorname{length}(E)_{j}^{(i)}}{\operatorname{rank} E} \end{array}$$

We can define the notion of:

- a  $\nu$ -parabolic Higgs bundle  $(E, \Phi, \{l_*^{(i)}\}_{1 \le i \le n})$  and
- $\bullet$  the  $\alpha$ -stability conditions for a  $\nu$ -parabolic Higgs bundle as in the same way above.

1.9. Moduli spaces of stable parabolic connections and stable parabolic Higgs bundles.

• Fix  $(C, \mathbf{t})$  and  $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)$ . We can define the moduli space of  $\boldsymbol{\alpha}$ -stable parabolic connections

(3) 
$$\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,n,d) = \{(E,\nabla,\{l^{(i)}_*\}_{1 \le i \le n})\} / \simeq$$

• Moreover for  $\boldsymbol{\nu} \in \mathcal{N}_{r,H}^n$ , we can define the moduli space of  $\boldsymbol{\alpha}$ -stable parabolic Higgs bundles:

•

(4) 
$$\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,n,d)_{H} = \{(E,\Phi,\{l^{(i)}_{*}\}_{1 \le i \le n})\}/\simeq .$$

1.10. Existence of algebraic moduli space of  $\alpha$ -stable  $\nu$ -parabolic connections.

Theorem 1.1. (Inaba-Iwasaki-Saito RIMS2006 [6], ASPM2006 [7], Inaba, JAG2013 [5]). There exists the relative fine moduli scheme  $\pi: \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/\tilde{M}_{g,n}\times\mathcal{N}^{n}_{r}(d)}(r,d,n) \longrightarrow \tilde{M}_{g,n}\times\mathcal{N}^{n}_{r}(d)$ such that  $\pi$  is smooth and quasi-projective.

**Corollary 1.1.** For fixed (*C*, t) and  $\boldsymbol{\nu} \in \mathcal{N}^n(d)$  the r

Corollary 1.1. For fixed  $(C, \mathbf{t})$  and  $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)$ , the moduli space  $\mathcal{M}_{(C, \mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, r, n, d)$ 

is a smooth quasi-projective algebraic scheme (most case irreducible) of dimension

$$2r^{2}(g-1) + nr(r-1) + 2 = 2N.$$

Moreover  $\mathcal{M}^{\pmb{\alpha}}_{(C,\mathbf{t})}(\pmb{\nu},r,n,d)$  admits the natural algebraic symplectic structure.

1.11. As in the similar way, we can obtain the existence of algebraic moduli space of  $\alpha$ -stable  $\nu$ -parabolic Higgs bundles (K(D)-pairs of Boden and Yokogawa).

Theorem 1.2. There exists the relative fine moduli scheme  $\pi: \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/\tilde{M}_{g,n}\times\mathcal{N}^{n}_{r}(d)}(r,d,n)_{H} \longrightarrow \tilde{M}_{g,n}\times\mathcal{N}^{n,H}_{r}$  such that  $\pi$  is smooth and quasi-projective.

Corollary 1.2. For fixed  $(C, \mathbf{t})$  and  $\boldsymbol{\nu} \in \mathcal{N}_{r,H}^n$ , the moduli space  $\mathcal{M}_{(C,\mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu},r,n,d)_H$ 

is a smooth quasi-projective algebraic scheme (most case variety) of dimension

$$2r^{2}(g-1) + nr(r-1) + 2 = 2N.$$

Moreover  $\mathcal{M}^{\pmb{\alpha}}_{(C,\mathbf{t})}(\pmb{\nu},r,n,d)_H$  admits the natural algebraic symplectic structure.

1.12. Example: Moduli space of connections, Painlevé VI case. Consider the case:  $C = \mathbf{P}^1, r = 2, n = 4, d = -1$  and a generic  $\boldsymbol{\nu} \in \mathcal{N}_2^4(-1)$ . We can normalize  $\mathbf{t} = \{t_1, t_2, t_3, t_4\} = \{0, 1, t, \infty\}$  and  $\boldsymbol{\nu} = \{\pm \nu_1, \pm \nu_2, \pm \nu_3.\nu_4, 1 - \nu_4\}$ . Then the moduli space  $M(\mathbf{t}, \boldsymbol{\nu}) = \mathcal{M}_{(\mathbf{P}^1, \mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, 2, 4, -1)$  is an algebraic surface. dim  $M(\mathbf{t}, \boldsymbol{\nu}) = 2N = 4(0 - 1) + 4 \times 2 + 2 = 2$ .  $M(\mathbf{t}, \boldsymbol{\nu})$  has a nice compactification  $S_{\mathbf{t}, \boldsymbol{\nu}} = \overline{M}(\mathbf{t}, \boldsymbol{\nu})$ .  $S_{\mathbf{t}, \boldsymbol{\nu}}$  is a 8-points blowing up of  $\Sigma_2 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))$ . The points of blowing up depends on the local exponents  $\boldsymbol{\nu}$ . See below. The anti-canonical divisor of  $S_{\mathbf{t}, \boldsymbol{\nu}}$  is given  $-K_{S_{\mathbf{t}, \boldsymbol{\nu}}} = 2Y_0 + Y_1 + Y_2 + Y_3 + Y_4$ .  $\overline{M(\mathbf{t}, \boldsymbol{\nu})} = S_{\mathbf{t}, \boldsymbol{\nu}} \setminus Y$ .



1.13. Example: Moduli space of parabolic Higgs bundles. Consider the case:  $C = \mathbf{P}^1, r = 2, n = 4, d = -1$  and a generic  $\boldsymbol{\nu}' \in \mathcal{N}_2^4(0)$ . We can normalize  $\mathbf{t} = \{t_1, t_2, t_3, t_4\} = \{0, 1, t, \infty\}$  and  $\boldsymbol{\nu}' = \{\pm \nu_1, \pm \nu_2, \pm \nu_3, \pm \nu_4\}$ . Then  $M(\mathbf{t}, \boldsymbol{\nu}')_H = \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathbf{P}^1, \mathbf{t})}(\boldsymbol{\nu}', 2, 4, -1)_H$  is also an algebraic surface. dim  $M_H(\mathbf{t}, \boldsymbol{\nu}') =$  $2N = 4(0 - 1) + 4 \times 2 + 2 = 2$ .  $M_H(\mathbf{t}, \boldsymbol{\nu}')$  has a nice compactification  $S_{\mathbf{t}, \boldsymbol{\nu}'} =$  $\overline{M(\mathbf{t}, \boldsymbol{\nu}')_H}$ .  $S_{\mathbf{t}, \boldsymbol{\nu}'}$  is a 8-points blowing up of  $\Sigma_2 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))$ .  $-K_{S_{\mathbf{t}, \boldsymbol{\nu}'}} =$  $2Y_0 + Y_1 + Y_2 + Y_3 + Y_4$ .  $\overline{M(\mathbf{t}, \boldsymbol{\nu}')_H} = S_{\mathbf{t}, \boldsymbol{\nu}'} \setminus Y$ . We can see that algebraic structures of  $M(\mathbf{t}, \boldsymbol{\nu})$  and  $M(\mathbf{t}, \boldsymbol{\nu}')_H$  are different.



### 2. The Riemann-Hilbert correspondence

2.1. Moduli space of representations of  $\pi_1(C \setminus D(\mathbf{t}), *)$ . Define:

$$\mathcal{RP}^r_{(C,\mathbf{t})} = \operatorname{Hom}(\pi_1(C \setminus D(\mathbf{t}), *), GL_r(\mathbf{C})) / / Ad(GL_r(\mathbf{C}))$$

or

$$\mathcal{RP}_{(C,\mathbf{t})}^{r,s} = \operatorname{Hom}(\pi_1(C \setminus D(\mathbf{t}), *), SL_r(\mathbf{C})) / / Ad(SL_r(\mathbf{C}))$$

By definition,  $\mathcal{RP}^{r}_{(C,\mathbf{t})}$  and  $\mathcal{RP}^{r,s}_{(C,\mathbf{t})}$  are affine varieties associated to the invariant ring of matrices.

Replacing  $T = \mathcal{M}'_{g,n}$  by a certain finite étale covering  $u: T' \longrightarrow T$  and varying  $((C, \mathbf{t}), \nu) \in T' \times \mathcal{N}^{(n)}_{r}(d)$  we can define a morphism (5)  $\mathbf{RH}: \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C}, \mathbf{t})/T'}(r, n, d) \longrightarrow \mathcal{RP}^{r}_{n,T'}$ 

which makes the diagram

(6)  

$$\begin{array}{cccc}
\mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/T'}(r,n,d) &\xrightarrow{\mathbf{RH}} & \mathcal{RP}^{r}_{n,T'} \\
& \Phi_{r,n,d} & & & \downarrow \phi^{r}_{n} \\
& & T' \times \mathcal{N}^{(n)}_{r}(d) &\xrightarrow{Id \times rh} & T' \times \mathcal{A}^{(n)}_{r}
\end{array}$$

commute.

## $2.2. \ {\rm Riemann-Hilbert\ correspondences}.$

**Theorem 2.1.** (Inaba-Iwasaki-Saito, RIMS2006 [6], ASPM2006[7], Inaba JAG2013[5] ). Assume that  $\alpha$  is generic. The Riemann-Hilbert correspondence

(7) 
$$\mathbf{RH}: \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/T'}(r,n,d) \longrightarrow \mathcal{RP}^{r}_{n,T'} \times_{\mathcal{A}^{(n)}_{r}} \mathcal{N}^{(n)}_{r}$$

is a proper surjective bimeromorphic analytic morphism. In particular, for each  $((C, \mathbf{t}), \boldsymbol{\nu}) \in T' \times \mathcal{N}_r^{(n)}(d)$ , the restricted morphism

(8) 
$$\mathbf{RH}_{((C,\mathbf{t}),\boldsymbol{\nu})}: \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\nu})}(r,n,d) \longrightarrow \mathcal{RP}^{r}_{(C,\mathbf{t}),\mathbf{a}}$$

gives an analytic resolution of singularities of  $\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$  where  $\mathbf{a} = rh(\boldsymbol{\nu})$  is a image of small Riemann-Hilbert correspondence rh.

#### 3. General schemes of the geometry of Riemann-Hilbert Correspondences

Consider the following diagram:

Theorem 3.1. If the Riemann-Hilbert map

$$\mathbf{RH}_{t,\boldsymbol{\nu}}: \tilde{M}_{t,\boldsymbol{\nu}} \longrightarrow \tilde{\mathcal{R}}_{t,\boldsymbol{\mu}(\boldsymbol{\nu})}$$

is a proper, surjective bimeromorphic holomorphic map for any  $(t, \boldsymbol{\nu}) \in \tilde{T} \times N$ . Then the corresponding isomonodromic differential equations satisifies the geometric Painlevé property.

## Isomonodromic Flows: $\nu$ Generic Case

The Riemann-Hilbert correspondence  $\mathbf{RH}_{\nu}$  induce an analytic isomorphisms for all  $t \in \tilde{T}_n$ . Pulling back the constant section on the right hand side, we have the isomonodromic flows on the left hand side. These isomondromic flows satisfy the Geometric Painlevé property.



FIGURE 1. Riemann-Hilbert correspondence and isomonodromic flows for generic  $\nu$ 

# **Isomonodromic Flows: Special Case**

If  $\nu$  is special (resonant, reducible), the right hand side have singularity. On the other hand, the left hand side is always nonsingular, hence  $\mathbf{RH}_{\nu}$  gives a simultaneous resolution of singularities. Riccati flows.



 $_{\rm Figure~2.}$  Riemann-Hilbert correspondence and isomonodromic flows for special  $\nu$ 

3.1. Geometric Painlevé property of the NDFE arrising from Isomonodromic deformation of LODE.

**Corollary 3.1.** ([6], [7], [5]) Differential equations arrising from isomonodromic deformations of linear connections with regular singularities over a curve satisfies the geometric Painlevé property.

**Remark 3.1.** We can extend the above result in the following cases;

- Connections of any rank with generic unramified irregular singularity on smooth projective curves. (Inaba-Saito, KJM2012 [9])
- Logarithmic connections of any rank with fixed spectral type with multiplicities. (Inaba-Saito, in preparation).

3.2. Moduli spaces of monodromy representations and generalized Stokes data related to Painlevé equations. Monodromy variety for Painlevé VI case

Define

$$\mathcal{RP}_4^{2,s} = \operatorname{Hom}(\pi_1(\mathbf{P}^1 \setminus \{t_1, t_2, t_3, t_4\}, SL(2, \mathbf{C})) / / Ad(SL_2(\mathbf{C})))$$
  
=  $\{(M_1, M_2, M_3, M_4) \in SL_2(\mathbf{C}), M_1M_2M_3M_4 = I_2\} / / Ad(SL_2(\mathbf{C})))$   
=  $\{(M_1, M_2, M_3) \in SL_2(\mathbf{C})\} / / Ad(SL_2(\mathbf{C})))$ 

We can describe the moduli space as follows. Take  $M_i \in SL_2(\mathbb{C})$  for i = 1, 2, 3 and set

$$a_i = \operatorname{Tr}[M_i], i = 1, 2, 3$$
  $a_4 := \operatorname{Tr}[M_4] = \operatorname{Tr}[M_4^{-1}] = \operatorname{Tr}[M_1 M_2 M_3]$ 

For a circle permutation (i, j, k) of (1, 2, 3), set

$$x_i = \operatorname{Tr}[M_j M_k].$$

Then the invariant ring is given by

$$\mathbf{C}[M_1, M_2, M_3]^{SL_2(\mathbf{C})} = \mathbf{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4] / (f(\mathbf{x}, \mathbf{a}))$$

where we set the cubic polynomial given by Fricke-Klein, Jimbo and Iwasaki.

$$f(\mathbf{x}, \mathbf{a}) = x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(\mathbf{a}) x_1 - \theta_2(\mathbf{a}) x_2 - \theta_3(\mathbf{a}) x_3 + \theta_4(\mathbf{a})$$

$$\begin{array}{rcl} \theta_i(\mathbf{a}) &=& a_i a_4 + a_j a_k, & (i,j,k) = \mathsf{a} \text{ cyclic permutation of } (1,2,3), \\ \theta_4(\mathbf{a}) &=& a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4. \end{array}$$

**Theorem 3.2.** The monodromy variety of Painlevé VI is isomorphic to the affine variety

$$\mathcal{X} = \mathcal{RP}_4^{2,s} = SL_2(\mathbf{C})^3 / Ad(SL_2(\mathbf{C}))$$
  
= Spec( $\mathbf{C}[x_1, x_2, x_3, a_1.a_2, a_3, a_4] / (f(\mathbf{x}, \mathbf{a}))$   
= {( $\mathbf{x}, \mathbf{a}$ )  $\in \mathbf{C}^7, f(\mathbf{x}, \mathbf{a}) = 0$ }  $\subset \mathbf{C}^7$ 

Moreover for a fixed  $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbf{C}^4$ 

$$\mathcal{X}_{\mathbf{a}} = \mathcal{RP}_{4,\mathbf{a}}^{2,s} = \operatorname{Spec}(\mathbf{C}[x_1, x_2, x_3] / (f(\mathbf{x}, \mathbf{a}))) = \mathcal{X}_{\mathbf{a}} \subset \mathbf{C}^3 \subset \mathbf{P}^3$$

The Riemann-Hilbert correspondence induces an analytic isomorphism for generic  $\boldsymbol{\nu} = (\pm \nu_i, i = 1, 2, 3, \nu_4, 1 - \nu_4)$ .  $a_i = 2\cos(-2\pi\nu_i)$ .

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\nu}}: M(\mathbf{t},\boldsymbol{\nu}) \xrightarrow{\simeq} \mathcal{X}_{\mathbf{a}}$$

For special  $\nu$ , we have a proper bimeromorphic analytic morphism (analytic resolution of singularities).

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\nu}}: M(\mathbf{t},\boldsymbol{\nu}) {\longrightarrow} \mathcal{X}_{\mathbf{a}}$$



### 4. Types of Singularities of Linear connetions

Let us list up the types of irregular singular points of lin. connetions of rank 2 on  $\mathbf{P}^1$  which induces iso-Stokes-Monodromy differential equations (=Lax equations)isomorphic to the Painlevé equations of the types in the table. This results follows from original result due to Garnier, Okamoto, Miwa-Jimbo-Ueno and Ohyama, Kawamuko, Sakai and Okamoto. (Moreover Flaschka and Newell obtained PII(FN).)

Dynkin	Painlevé equation	s(0)	s(1)	$s(\infty)$	s(t)	no. of parameters
$ ilde{D}_4$	PVI	0	0	0	0	4
$\tilde{D}_5$	PV	0	0	1	-	3
$\tilde{D}_6$	deg $PV = PIII(D6)$	0	0	1/2	-	2
$\tilde{D}_6$	PIII(D6)	1	-	1	-	2
$\tilde{D}_7$	PIII(D7)	1/2	-	1	-	1
$\tilde{D}_8$	PIII(D8)	1/2	-	1/2	-	0
$ ilde{E}_6$	PIV	0	-	2	-	2
$ ilde{E}_7$	PII(FN)=PII	0	-	3/2	-	1
$ ilde{E}_7$	PII	-	-	3	-	1
$\tilde{E}_8$	PI	-	-	5/2	-	0

TABLE 1. The type of singularities for linear problems and Pailevé equations

Equations of Moduli space of Stokes-Monodromy data

The following result is due to a joint work with Marius van der Put ([21]).

4.1. Family (-, -, 5/2) and Painlevé PI. According to Definition and examples 1.10, a differential module of this type need not have a solution for the strong Riemann-Hilbert problem. We deal here with the modules for which there is a solution, i.e., are represented by a matrix differential equation  $\frac{d}{dz} + A_0 + A_1 z + A_2 z^2$  with nilpotent  $A_2$  which can be normalized into  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . The map  $z \mapsto \lambda z + \mu$  is used to normalize the eigenvalues at  $\infty$  to  $\pm (z^{5/2} + \frac{t}{2} \cdot z^{1/2})$ . Conjugation with a constant matrix of the form  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  leads to the normalization

$$\frac{d}{dz} + \left(\begin{array}{cc} p & t+q^2 \\ -q & -p \end{array}\right) + \left(\begin{array}{cc} 0 & q \\ 1 & 0 \end{array}\right) z + \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right) z^2 \ .$$

The space AnalyticData is given by the formal monodromy and 5 Stokes maps which are on a basis  $e_1, e_2$  of the formal solution space at  $\infty$  given by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_5 & 1 \end{pmatrix}.$$

Their product is the topological monodromy and thus equal to  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . The base change  $e_1, e_2 \mapsto \lambda e_1, \lambda e_2$  does not effect these matrices. Hence the coordinate ring of  $\mathcal{R}$  is generated by  $a_1, \ldots, a_5$  and their relations are given by the above matrix identity.

After eliminating  $a_2$  by  $a_2 = 1 + a_4 a_5$  and  $a_1$  by  $a_1 = -1 - a_3 a_4$ , one obtains for the remaining variables  $a_3, a_4, a_5$  just one equation and  $\mathcal{R}$  is a non singular affine cubic surface with three lines at infinity, given by  $a_3 a_4 a_5 + a_3 + a_5 + 1 = 0$ .

4.2. Family (-, -, 5/2) and Painlevé I,  $PI(\tilde{E}_8)$ . The family of connection with the data can be written as

The singular points $z$	$\infty$			
Katz invariant	$\frac{5}{2}$			
generalized local exponents	$\pm (z^{5/2} + \frac{t}{2}z^{1/2})$			

(9) 
$$\nabla_{\frac{d}{dz}} = \frac{d}{dz} + A_0 + zA_1 + z^2A_2 = \frac{d}{dz} + A, \text{ where}$$
(10) 
$$A_0 = \begin{pmatrix} p & q^2 + t \\ -q & -p \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} p & q^2 + zq + z^2 + t \\ z - q & -p \end{pmatrix},$$

$$\nabla_{\frac{d}{dt}} = \frac{d}{dt} + B$$

$$B := B_0 + zB_1, \quad B_0 = \begin{pmatrix} 0 & 2q \\ 1 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

## Painlevé I, $PI(\tilde{E}_8)$

(11) 
$$\begin{cases} \frac{dq}{dt} = 2p\\ \frac{dp}{dt} = 3q^2 + t \end{cases}$$

The system (11) is equivalent to the following second order equation.

(12) 
$$q'' = 6q^2 + 2t$$

(13) 
$$\Omega = dp \wedge dq - dH_{IE_8} \wedge dt, \ H_{IE_8}(p,q,t,\theta) = -p^2 + q^3 + tq$$

Equation (11) is equivalent to the following Hamiltonian system:

(14) 
$$\begin{cases} \frac{dq}{dt} = -\frac{\partial H_{IE_8}}{\partial p}, \\ \frac{dp}{dt} = \frac{\partial H_{IE_8}}{\partial q}. \end{cases}$$

5. Apparent singularities (a joint work with S. Szabo)

5.1. Apparent singularities of connections and Higgs bundles.

- $C, \mathbf{t}$  as before.
- We set  $L = \Omega_C^1(t_1 + \dots + t_n)$ . We assume that  $n \ge 1$  and  $\deg L = 2g 2 + n > 0$ .

Consider the moduli spaces

(15) 
$$M_{DR}(\boldsymbol{\nu}) = \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,n,d) = \{(E,\nabla,\{l^{(i)}_*\}_{1 \le i \le n})\}/\simeq .$$
  
(16)

$$M_{H}(\boldsymbol{\nu}) = \mathcal{M}_{(C,\mathbf{t})}^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, r, n, d)_{H} = \{(E, \Phi, \{l_{*}^{(i)}\}_{1 \le i \le n})\} / \simeq$$

For simplicity, we assume that  $\nu \in \mathcal{N}_r^n(d)$  or  $\nu \in \mathcal{N}_{r,H}^n$  are non-resonant and so generic such that all members of moduli spaces are irreducible.

**Proposition 5.1.** Assume that  $\exists \sigma \in H^0(C, E) \setminus \{0\}$  and  $\deg L = 2g - 2 + n \ge 1$  and  $\deg D = n \ge 1$ . Moreover assume that  $(E, \nabla)$  (resp.  $(E, \Phi)$ ) is irreducible. Set

(17) 
$$F = \bigoplus_{j=0}^{r-1} L^{-j} = \mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(r-1)}.$$

 $\exists$  a natural embedding  $F \hookrightarrow E$  such that  $H^0(C, F) \simeq \mathbf{C}\sigma \subset H^0(C, E)$ . Define the torsion sheaf  $T_A$  by the exact sequence

(18) 
$$0 \longrightarrow F \longrightarrow E \longrightarrow T_A \longrightarrow 0,$$

Then

length 
$$T_A = d - r(g - 1) + r^2(g - 1) + n \frac{r(r - 1)}{2}$$
.

Definition 5.1. For an irreducible parabolic connection  $(E, \nabla, l)$ (resp. irreducible parabolic Higgs bundles  $(E, \Phi, l)$ ) and a non-zero section  $\sigma$ , we call the support of  $T_A$  apparent singular points of the parabolic connection  $(E, \nabla, l)$  (resp.  $(E, \Phi, l)$ ) with the cyclic vector  $\sigma$ . Now assume that deg E = d = r(g-1)+1. We have dim  $H^0(C, E) = \dim H^1(C, E) + 1$  by Riemann-Roch. If moreover  $H^1(C, E) = 0$ , we have a non-zero section  $\sigma \in H^0(C, E) \simeq \mathbf{C}\sigma$  unique up to non-zero scalar multiplications.

Theorem 5.1. Under the same notation and assumption as before, let us assume that

(19) 
$$d = \deg E = r(g-1) + 1,$$

(20) 
$$H^1(C, E) = 0.$$

Then we have a natural unique embedding  $F \hookrightarrow E$  which yields

(21) 
$$0 \longrightarrow F \longrightarrow E \longrightarrow T_A \longrightarrow 0.$$

Then the sheaf  $T_A$  is a torsion sheaf of length

(22) 
$$N = r^2(g-1) + n\frac{r(r-1)}{2} + 1.$$

5.2. The case of parabolic Higgs bundles.

- Let  $(E, \Phi, l)$  be the  $\nu$ -parabolic Higgs bundles of degree  $d = \deg E = r(g-1)+1$  and assume that  $\dim H^0(C, E) = 1$ . Again we set  $L = \Omega^1_C(D)$ .
- We have a canonical exact sequence

$$\begin{array}{c} 0 \longrightarrow F \longrightarrow E \longrightarrow T \longrightarrow 0\\ \text{with } F = \oplus_{j=1}^{r} L^{-(j-1)} \text{ and with apparent singularities}\\ \mathrm{supp} T = \{q_1, \cdots, q_N\} \end{array}$$

where

$$N = r^2(g-1) + n\frac{r(r-1)}{2} + 1 = \frac{1}{2}\dim M_H(\boldsymbol{\nu})$$

5.2.1. Spectral curves. Let

$$p: \mathbf{P} = \mathbb{P}(\mathcal{O}_C \oplus L^{-1}) \longrightarrow C$$

be the  $\mathbb{P}^1$ -bundle over C which is a relative compactification of the total space of  $L \longrightarrow C$ . The canonical section  $x \in H^0(P, \mathcal{O}_P(1) \otimes p^*(L))$  can be used to define the spectral curve

$$C_s : \det(xI_r - \Phi) = x^r - s_1 x^{r-1} - s_2 x^{r-2} - \dots + s_r = 0 \subset L \subset P$$

with the natural map  $\pi : C_s \longrightarrow C$  and  $s_i \in H^0(C, L^i)$ .



FIGURE 3. The ruled surface and the curve



**Proposition 5.2.** [BNR, [3]]. Assume that  $C_s$  is a smooth and irreducible Then there exists one to one correspondence

$$(E, \Phi, l) \Leftrightarrow (\pi : C_s \longrightarrow C, \xi)$$

where  $\xi$  is a line bundle on  $C_s$ . The correspondence  $\Leftarrow$  is given by  $\pi_*\xi = E$  and the structure of  $\pi_*\mathcal{O}_{C_s}$ -algebra.

Since  $H^0(C_s,\xi) = H^0(C,E) = \mathbf{C}$ , we see that a unique nonzero effective diviosr  $\delta$  such that

$$\mathcal{O}_C(\delta) \simeq \xi$$

of degree

$$\deg \delta = \deg \xi = \deg E - \deg F = r(g-1) + 1 + (2g-2+n)\frac{r(r-1)}{2} = N.$$

We have the natural exact sequence

$$0 \longrightarrow \mathcal{O}_{C_s} \longrightarrow \xi \longrightarrow \tilde{T} \longrightarrow 0$$
$$0 \longrightarrow \pi_* \mathcal{O}_{C_s} \longrightarrow \pi_* \xi \longrightarrow \pi_* \tilde{T} \longrightarrow 0$$

and  $\pi_* \mathcal{O}_{C_s} \simeq F$ ,  $\pi_* \xi = E$  and  $\pi_* \tilde{T} = T$ .  $0 \longrightarrow F \longrightarrow E \longrightarrow T \longrightarrow 0$ 

5.3. Higgs case. For  $(E, \Phi, l)$ , take the data of spectral curve and the line bundle  $(\pi : C_s \longrightarrow C, \xi)$ .

Since  $H^0(C, E)$  a nonzero section  $\sigma$ , there exist a non-zero section  $\tilde{\sigma} \in H^0(C_s, \xi)$  such that  $\pi_*(\tilde{\sigma}) = \sigma$ . Let  $\delta = p_1 + \cdots + p_N$  be the zero divisor of  $\tilde{\sigma}$ . We have the exact sequence of sheaves on  $C_s$ 

$$0 \longrightarrow \mathcal{O}_{C_s} \xrightarrow{\tilde{\sigma}} \mathcal{O}_{C_s}(\delta) \longrightarrow T_{\delta} \longrightarrow 0$$

The pushforward of this sequence

$$0 \longrightarrow \pi_* \mathcal{O}_{C_s} \longrightarrow \pi_* \xi \longrightarrow \pi_* T_\delta \longrightarrow 0$$

is isomorphic to

$$0 \longrightarrow F \longrightarrow E \longrightarrow T \longrightarrow 0$$

So we have

$$\pi(\delta) = \sum_{i=1}^{N} \pi(p_i) = \sum_{i=1}^{N} q_i.$$

$$0 \longrightarrow F \longrightarrow E \longrightarrow T \longrightarrow 0$$

$$\downarrow \Phi \qquad \downarrow \Phi \qquad \downarrow \oplus \Phi q_i$$

$$0 \longrightarrow F \otimes L \longrightarrow E \otimes L \longrightarrow T \otimes L \longrightarrow 0$$
The dual coordinates  $\{p_1, \dots p_N\}.$ 

$$p_i = \Phi(q_i) \in L_{i}$$

$$p_i = \Phi(q_i) \in L_{q_i}$$



FIGURE 5. A Spectral curve and a spectral divisor

5.4. Geometric aspects of Higgs cases. Assume that  $\nu$  is generic so that all members  $(E, \Phi, l) \in M_H(\nu)$  are irreducible.

$$M_H(\boldsymbol{\nu})^0 = \{ (E, \Phi, l), \deg E = r(g-1) + 1, H^0(C, E) \simeq \mathbf{C} \}.$$

Then we have the following

$$\begin{array}{ll} M_{H}(\boldsymbol{\nu})^{0} &\simeq M^{0} \coloneqq \{(C_{s}, \boldsymbol{\xi} = \mathcal{O}_{C}(\boldsymbol{\delta}))\} \xrightarrow{\phi} \operatorname{Hilb}^{N}(L) \\ \downarrow & \downarrow \\ \operatorname{Hilb}^{N}(C) & |C_{s}| & \operatorname{Hilb}^{N}(C) \end{array}$$

$$\begin{array}{l} \operatorname{Apparent\ map} & \operatorname{Hitchin\ fibration} \end{array}$$

 $\phi((C_s,\xi))=I_\delta: \text{Ideal sheaf of }\delta\subset C_s\subset L$ 

In many known cases, we can check that

 $\phi$  is a dominant birational morphism, and we expect that this statement is always true.

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