Some results on the Yang-Mills flow and its application

Min-Chun Hong
The University of Queensland
Brisbane, Australia

10 August 2016

Suppose that M is a compact four dimensional Riemannian manifold and E is a vector bundle over M.
For each connection D_A, the Yang-Mills functional is defined by

$$\mathrm{YM}(A; M) = \int_M |F_A|^2 \, dv,$$

where F_A is the curvature of D_A.

Suppose that M is a compact four dimensional Riemannian manifold and E is a vector bundle over M.
For each connection D_A, the Yang-Mills functional is defined by

$$\mathrm{YM}(A; M) = \int_M |F_A|^2 \, dv,$$

where F_A is the curvature of D_A.

In local trivialization,

$$D_A = d + A, \quad F_A = dA + A \wedge A.$$

where $A \in \Gamma(\mathrm{End}E \otimes T^*M)$ is the connection matrix one form.

Suppose that M is a compact four dimensional Riemannian manifold and E is a vector bundle over M.
For each connection D_A, the Yang-Mills functional is defined by

$$\mathrm{YM}(A;M)=\int_M |F_A|^2\,dv,$$

where F_A is the curvature of D_A.

In local trivialization,

$$D_A=d+A,\quad F_A=dA+A\wedge A.$$

where $A\in\Gamma(\mathrm{End}E\otimes T^*M)$ is the connection matrix one form.

We say that a connection D_A is a *Yang-Mills connection* if D_A satisfies the Yang-Mills equation

$$D_A^*F_A=0\,. \tag{1}$$

Yang-Mills equations

- Yang-Mills equations originated from the theory of classical fields in particle physics. Self-dual Yang-Mills connections are called ‘instantons’ in physics.

Yang-Mills equations

- Yang-Mills equations originated from the theory of classical fields in particle physics. Self-dual Yang-Mills connections are called 'instantons' in physics.
- Atiyah, Hitchin, Drinfel'd and Manin in 1978 established the fundamental existence result of instantons on S^4.

Yang-Mills equations

- Yang-Mills equations originated from the theory of classical fields in particle physics. Self-dual Yang-Mills connections are called 'instantons' in physics.
- Atiyah, Hitchin, Drinfel'd and Manin in 1978 established the fundamental existence result of instantons on S^4.
- Uhlenbeck (CMP 1982) proved the removal singularity theorem for Yang-Mills connections on 4-manifolds.

Yang-Mills equations

- Yang-Mills equations originated from the theory of classical fields in particle physics. Self-dual Yang-Mills connections are called 'instantons' in physics.
- Atiyah, Hitchin, Drinfel'd and Manin in 1978 established the fundamental existence result of instantons on S^4.
- Uhlenbeck (CMP 1982) proved the removal singularity theorem for Yang-Mills connections on 4-manifolds.
- Uhlenbeck (CMP 1982) used a gauge fixing theorem on 4-manifolds to establish a weak compact theorem.

Yang-Mills equations

- Yang-Mills equations originated from the theory of classical fields in particle physics. Self-dual Yang-Mills connections are called ‘instantons’ in physics.
- Atiyah, Hitchin, Drinfel’d and Manin in 1978 established the fundamental existence result of instantons on S^4.
- Uhlenbeck (CMP 1982) proved the removal singularity theorem for Yang-Mills connections on 4-manifolds.
- Uhlenbeck (CMP 1982) used a gauge fixing theorem on 4-manifolds to establish a weak compact theorem.
- Donaldson (JDG 1983) successfully applied Yang-Mills theory to four dimensional geometric topology.

The Yang-Mills flow

The Yang-Mills flow equation is

$$\frac{\partial D_A}{\partial t} = -D_A^* F_A \tag{2}$$

with $D_A(0) = D_0$, where D_0 is a given smooth connection.

The Yang-Mills flow

The Yang-Mills flow equation is

$$\frac{\partial D_A}{\partial t} = -D_A^* F_A \qquad (2)$$

with $D_A(0) = D_0$, where D_0 is a given smooth connection.

- Atiyah and Bott (Phil. Trans. Roy. Soc. London A 1982) suggested to use the method of the Yang-Mills flow to establish the existence of Yang-Mills connections.

The Yang-Mills flow

The Yang-Mills flow equation is

$$\frac{\partial D_A}{\partial t} = -D_A^* F_A \tag{2}$$

with $D_A(0) = D_0$, where D_0 is a given smooth connection.

- Atiyah and Bott (Phil. Trans. Roy. Soc. London A 1982) suggested to use the method of the Yang-Mills flow to establish the existence of Yang-Mills connections.
- Donaldson (Proc. Lond. Math. Soc. 1985) proved the global existence of Yang-Mills flow and used the flow to prove that an irreducible holomorphic vector bundle E over a compact Kähler surface X admits a unique Hermitian-Einstein connection if and only if it is stable.

The Yang-Mills flow

The Yang-Mills flow equation is

$$\frac{\partial D_A}{\partial t} = -D_A^* F_A \tag{2}$$

with $D_A(0) = D_0$, where D_0 is a given smooth connection.

- Atiyah and Bott (Phil. Trans. Roy. Soc. London A 1982) suggested to use the method of the Yang-Mills flow to establish the existence of Yang-Mills connections.
- Donaldson (Proc. Lond. Math. Soc. 1985) proved the global existence of Yang-Mills flow and used the flow to prove that an irreducible holomorphic vector bundle E over a compact Kähler surface X admits a unique Hermitian-Einstein connection if and only if it is stable.
- Uhlenbeck-Yau (CPAM 1986) established the result in the case of holomorphic vector bundles over compact Kähler manifolds, which is now called the Donaldson-Uhlenbeck-Yau theorem

Moreover, the Donaldson-Uhlenbeck-Yau theorem has been generalized by many authors.

- Simpson (JAMS 1988) generalized the Donaldson-Uhlenbeck-Yau theorem in holomorphic bundles E over non-compact manifolds X satisfying certain assumptions.

Moreover, the Donaldson-Uhlenbeck-Yau theorem has been generalized by many authors.

- Simpson (JAMS 1988) generalized the Donaldson-Uhlenbeck-Yau theorem in holomorphic bundles E over non-compact manifolds X satisfying certain assumptions.
- Hong (Ann. Global Anal. Geom. 2001) generalized the result to the Yang-Mills-Higgs flow in holomorphic vector bundles over compact Kähler manifolds.

Moreover, the Donaldson-Uhlenbeck-Yau theorem has been generalized by many authors.

- Simpson (JAMS 1988) generalized the Donaldson-Uhlenbeck-Yau theorem in holomorphic bundles E over non-compact manifolds X satisfying certain assumptions.
- Hong (Ann. Global Anal. Geom. 2001) generalized the result to the Yang-Mills-Higgs flow in holomorphic vector bundles over compact Kähler manifolds.
- Ni (Trans. Amer. Math. 2001) generalized the result to the Yang-Mills flow in holomorphic vector bundles over some complete Kähler manifolds.

Moreover, the Donaldson-Uhlenbeck-Yau theorem has been generalized by many authors.

- Simpson (JAMS 1988) generalized the Donaldson-Uhlenbeck-Yau theorem in holomorphic bundles E over non-compact manifolds X satisfying certain assumptions.
- Hong (Ann. Global Anal. Geom. 2001) generalized the result to the Yang-Mills-Higgs flow in holomorphic vector bundles over compact Kähler manifolds.
- Ni (Trans. Amer. Math. 2001) generalized the result to the Yang-Mills flow in holomorphic vector bundles over some complete Kähler manifolds.
- Later, X. Zhang (Canad. J. Math. 2005) generalized the result to the Yang-Mills-Higgs flow in holomorphic vector bundles over some complete Kähler manifolds.

In the case of holomorphic vector bundles are not stable,

- Hong and Tian (Math. Ann. 2004) established asymptotic behaviour of the Yang-Mills flow and proved the existence of a singular Hermitian Yang-Mills connection in a holomorphic vector bundle E over a Kähler manifold X, where the Hermitian Yang-Mills connection is smooth of codimension 2.

In the case of holomorphic vector bundles are not stable,

- Hong and Tian (Math. Ann. 2004) established asymptotic behaviour of the Yang-Mills flow and proved the existence of a singular Hermitian Yang-Mills connection in a holomorphic vector bundle E over a Kähler manifold X, where the Hermitian Yang-Mills connection is smooth of codimension 2.
- Using the Yang-Mills flow, Daskalopous and Wentworth (2004) settled a conjecture of Bando and Siu on the relation between the limiting Yang-Mills flow and the Harder-Narashimhan filtration on Kähler surface.

In the case of holomorphic vector bundles are not stable,

- Hong and Tian (Math. Ann. 2004) established asymptotic behaviour of the Yang-Mills flow and proved the existence of a singular Hermitian Yang-Mills connection in a holomorphic vector bundle E over a Kähler manifold X, where the Hermitian Yang-Mills connection is smooth of codimension 2.
- Using the Yang-Mills flow, Daskalopous and Wentworth (2004) settled a conjecture of Bando and Siu on the relation between the limiting Yang-Mills flow and the Harder-Narashimhan filtration on Kähler surface.
- Recently, by using this flow, Jacob (2015) and Sibley (2015) settled the conjecture of Bando and Siu on Kähler manifolds.

Without the holomorphic structure of the bundle E, it is a very interesting question to establish the global existence of the Yang-Mills flow on Riemannian manifolds.

Without the holomorphic structure of the bundle E, it is a very interesting question to establish the global existence of the Yang-Mills flow on Riemannian manifolds.

- In some special cases, Schlatter, Struwe and Tahvildar-Zadeh (Amer. J. Math. 1998) proved the global existence of the $SO(4)$-equivariant Yang-Mills flow on $\mathbb{R}^4$.

Without the holomorphic structure of the bundle E, it is a very interesting question to establish the global existence of the Yang-Mills flow on Riemannian manifolds.

- In some special cases, Schlatter, Struwe and Tahvildar-Zadeh (Amer. J. Math. 1998) proved the global existence of the $SO(4)$-equivariant Yang-Mills flow on $\mathbb{R}^4$.
- Hong and Tian (CAG 2004) proved the global existence of the m-equivariant Yang-Mills flow on $\mathbb{R}^4$.

Without the holomorphic structure of the bundle E, it is a very interesting question to establish the global existence of the Yang-Mills flow on Riemannian manifolds.

- In some special cases, Schlatter, Struwe and Tahvildar-Zadeh (Amer. J. Math. 1998) proved the global existence of the $SO(4)$-equivariant Yang-Mills flow on $\mathbb{R}^4$.
- Hong and Tian (CAG 2004) proved the global existence of the m-equivariant Yang-Mills flow on $\mathbb{R}^4$.
- Recently, Waldron [**?**] established global existence of the smooth solution to the Yang-Mills flow when $\|F_{A_0}^+\|_{L^2(M)}$ is sufficiently small.

Without the holomorphic structure of the bundle E, it is a very interesting question to establish the global existence of the Yang-Mills flow on Riemannian manifolds.

- In some special cases, Schlatter, Struwe and Tahvildar-Zadeh (Amer. J. Math. 1998) proved the global existence of the $SO(4)$-equivariant Yang-Mills flow on $\mathbb{R}^4$.
- Hong and Tian (CAG 2004) proved the global existence of the m-equivariant Yang-Mills flow on $\mathbb{R}^4$.
- Recently, Waldron [?] established global existence of the smooth solution to the Yang-Mills flow when $\|F^+_{A_0}\|_{L^2(M)}$ is sufficiently small.

In general cases, it is still open whether the Yang-Mills flow in four dimensional manifolds develop a singularity at finite time.

The weak solution of the Yang-Mills flow

Struwe (CVPDE 1994) proved the existence of a weak solution to the Yang-Mills flow in vector bundles on four manifolds with initial value in H^1, where the weak solution is gauge-equivalent to a smooth solution of the flow on $M \times (0, T)$ for a maximal existence time $T > 0$.

The weak solution of the Yang-Mills flow

Struwe (CVPDE 1994) proved the existence of a weak solution to the Yang-Mills flow in vector bundles on four manifolds with initial value in H^1, where the weak solution is gauge-equivalent to a smooth solution of the flow on $M \times (0, T)$ for a maximal existence time $T > 0$.
Moreover, as $t \to T$ the solution $A(t)$ converges, up-to gauge transformations, to a connection $A(T)$, smoothly away from at most finitely many points. Schlatter [**?**] did some blow-up analysis of the Yang-Mills flow at the singular time T, but there is no result concerning the energy identity of the Yang-Mills flow at the time T.

Recently, Hong and Schabrun (Preprint) establish an energy identity for the Yang-Mills flow as follows:

Theorem 1

Let $A(t)$ be a solution to the Yang-Mills flow in $M \times [0, T)$, where $T \in (0, \infty]$ is the maximal existence time, and $A(t)$ converges weakly as $t \to T$ to a connection $A(T)$. Then there are a finite number of bubble bundles $E_1, \cdots, E_l$ over S^4 and Yang-Mills connections $\tilde{A}_{1,\infty}, \cdots, \tilde{A}_{l,\infty}$ such that

$$\lim_{t \to T} YM(A(t)) = YM(A(T)) + \sum_{i=1}^{l} YM(\tilde{A}_{i,\infty}).$$

The Yang-Mills α-functional

Following a similar strategy of Sacks and Uhlenbeck for harmonic maps, Hong, Tian and Yin (CMH 2015) consider an α-Yang-Mills functional

$$\mathrm{YM}_\alpha(A; M) = \int_M (1 + |F_A|^2)^\alpha dv$$

for any $\alpha > 1$.

The Yang-Mills α-functional

Following a similar strategy of Sacks and Uhlenbeck for harmonic maps, Hong, Tian and Yin (CMH 2015) consider an α-Yang-Mills functional

$$\mathrm{YM}_\alpha(A; M) = \int_M (1 + |F_A|^2)^\alpha dv$$

for any $\alpha > 1$.

$D_A = D_0 + A$ is a critical point of the α-Yang-Mills functional if it satisfies the Euler-Lagrange equation

$$D_A^* \left((1 + |F_A|^2)^{\alpha - 1} F_A \right) = 0, \tag{3}$$

Yang-Mills α-flow

Hong, Tian and Yin introduced the Yang-Mills α-flow

$$\frac{\partial A}{\partial t} = -D_A^* F_A + (\alpha - 1)\frac{*(d|F_A|^2 \wedge *F_A)}{1 + |F_A|^2} \tag{4}$$

with initial condition $A(0) = A_0$.

Hong, Tian and Yin introduced the Yang-Mills α-flow

$$\frac{\partial A}{\partial t} = -D_A^* F_A + (\alpha - 1)\frac{*(d|F_A|^2 \wedge *F_A)}{1 + |F_A|^2} \tag{4}$$

with initial condition $A(0) = A_0$.

- the Yang-Mills α-flow admits a global smooth solution, whose limit set contains a smooth critical point of the Yang-Mills α-functional.

Yang-Mills α-flow

Hong, Tian and Yin introduced the Yang-Mills α-flow

$$\frac{\partial A}{\partial t} = -D_A^* F_A + (\alpha - 1)\frac{*(d|F_A|^2 \wedge *F_A)}{1 + |F_A|^2} \tag{4}$$

with initial condition $A(0) = A_0$.

- the Yang-Mills α-flow admits a global smooth solution, whose limit set contains a smooth critical point of the Yang-Mills α-functional.
- By considering the limit $\alpha \to 1$, the authors were then able to obtain existence results for Yang-Mills connections and its flow.

Yang-Mills α-flow

Hong, Tian and Yin introduced the Yang-Mills α-flow

$$\frac{\partial A}{\partial t} = -D_A^* F_A + (\alpha - 1)\frac{*(d|F_A|^2 \wedge *F_A)}{1 + |F_A|^2} \tag{4}$$

with initial condition $A(0) = A_0$.

- the Yang-Mills α-flow admits a global smooth solution, whose limit set contains a smooth critical point of the Yang-Mills α-functional.
- By considering the limit $\alpha \to 1$, the authors were then able to obtain existence results for Yang-Mills connections and its flow.
- Use the Yang-Mills α-flow to modify a new minimizing sequence, which converges to the same limit in the smooth topology up to gauge transformation away from finite singular points, which improved the Sedlacek result (CMP 1982).

Uhlenbeck's compactness theorem

By the gauge transformation S, a connection $D_A = d + A$ can be transformed to a new connection

$$\bar{D}_A = S^*(D_A) = S^{-1} \circ D_A \circ S = d + S^{-1} dS + S^{-1} AS,$$

we have

$$F_{\bar{D}_A} = S^{-1} F_A S.$$

Uhlenbeck's compactness theorem

By the gauge transformation S, a connection $D_A = d + A$ can be transformed to a new connection

$$\bar{D}_A = S^*(D_A) = S^{-1} \circ D_A \circ S = d + S^{-1}dS + S^{-1}AS,$$

we have

$$F_{\bar{D}_A} = S^{-1} F_A S.$$

Uhlenbeck established the fundamental compactness results that for a sequence of connections A_i in E over M with a uniform bound of $YM(A_i; M)$, there exists a subsequence A_i, a sequence of gauge transformation S_i^* and a finite set of singularities $\{x_l\}_{l=1}^N$ such that $S_i^*(D_{A_i})$ weakly converges to D_A in $H^1(M \backslash \{x_l\}_{l=1}^N)$.

Uhlenbeck's compactness theorem

By the gauge transformation S, a connection $D_A = d + A$ can be transformed to a new connection

$$\bar{D}_A = S^*(D_A) = S^{-1} \circ D_A \circ S = d + S^{-1} dS + S^{-1} AS,$$

we have

$$F_{\bar{D}_A} = S^{-1} F_A S.$$

Uhlenbeck established the fundamental compactness results that for a sequence of connections A_i in E over M with a uniform bound of $YM(A_i; M)$, there exists a subsequence A_i, a sequence of gauge transformation S_i^* and a finite set of singularities $\{x_l\}_{l=1}^N$ such that $S_i^*(D_{A_i})$ weakly converges to D_A in $H^1(M \backslash \{x_l\}_{l=1}^N)$.

Theorem 2

(Uhlenbeck's gauging fixing theorem) Assume that there exist a sufficiently small ε_0 and a positive r_0 such that

$$\int_{B_{r_0}(x_0)} |F_A|^2 dv \leq \varepsilon_0.$$

Then there is a gauge transformations $S = e^u$ and a new connection $D_a = S^*(D_A) = d + a$ such that

$$d^* a = 0 \quad in\ B_{r_0}(x_0), \quad a \cdot \nu = 0\ on\ \partial B_{r_0}(x_0),$$

satisfying

$$\int_{B_{r_0}(x_0)} \frac{1}{r_0^p} |a(t)|^p + |\nabla a(t)|^p \, dx \leq C \int_{B_{r_0}(x_0)} |F_{a(t)}|^p \, dx$$

for $2 \leq p < 4$.

A parabolic gauging fixing theorem

From now on, let $D_A = d + A$ be a smooth solution of the Yang-Mills flow in $M \times [0, t_1]$ for some $t_1 > 0$. i.e.

$$\frac{\partial D_A}{\partial t} = -D_A^* F_A \tag{5}$$

with initial condition $D_A(0) = D_0$, where D_0 is a given connection on E.

More recently, I established a parabolic gauge fixing theorem for Yang-Mills flow. More precisely, we have:

Theorem 3

Assume that there exist a sufficiently small ε_0 *and a positive* r_0 *such that* $\sup_{0\le t\le t_1}\sup_{x_0\in M}\int_{B_{r_0}(x_0)}|F_A(x,t)|^2 dv \le \varepsilon_0$. *Then there are a gauge transformations* $S(t) = e^{u(t)}$ *and a new connection* $D_a = S^*(D_A) = d + a$ *satisfying*

$$\frac{\partial a}{\partial t} = -D_a^* F_a + D_a s, \quad \text{in } B_{r_0}(x_0)\times[0,t_1), \tag{6}$$

$$d^* a(t) = 0 \quad \text{in } B_{r_0}(x_0), \quad a(t)\cdot\nu = 0 \text{ on } \partial B_{r_0}(x_0),$$

$$\int_{B_{r_0}(x_0)} \frac{1}{r_0^p}|a(t)|^p + |\nabla a(t)|^p\, dx \le C\int_{B_{r_0}(x_0)} |F_{a(t)}|^p\, dx$$

$$\int_0^{t_1}\int_{B_{r_0}(x_0)} |D_a s|^2 + |\frac{\partial a}{\partial t}|^2\, dx\, dt \le C\int_0^{t_1}\int_{B_{r_0}(x_0)} |\nabla_a F_a|^2\, dx\, dt,$$

for $2\le p<4$ *and all* $t\in[0,t_1]$, *where* $s(t) = S^{-1}(t)\circ\frac{d}{dt}S(t)$.

As an application of Theorem 3, we have

Theorem 4

Let D_{A_i} be a sequence of smooth solutions of the Yang-Mills flow (5) in $M \times [0, T_i)$ for $T_i \geq T$ with smooth initial values $A_i(0) \in H^1$, where $A_i(0)$ strongly converges to A_0 in H^1. Then, the solution D_{A_i} converges to a connection D_A, which is a weak solution of the Yang-Mills flow in $M \times [0, T]$ with initial value A_0. The weak solution of the Yang-Mills flow in $M \times [0, T]$ with initial value A_0 in H^1 is smooth for $0 < t \leq T$.

As an application of Theorem 3, we have

Theorem 4

Let D_{A_i} be a sequence of smooth solutions of the Yang-Mills flow (5) in $M \times [0, T_i)$ for $T_i \geq T$ with smooth initial values $A_i(0) \in H^1$, where $A_i(0)$ strongly converges to A_0 in H^1. Then, the solution D_{A_i} converges to a connection D_A, which is a weak solution of the Yang-Mills flow in $M \times [0, T]$ with initial value A_0. The weak solution of the Yang-Mills flow in $M \times [0, T]$ with initial value A_0 in H^1 is smooth for $0 < t \leq T$.

Remark: Theorem 7 provide a new proof of the local existence of the Yang-Mills flow with initial value $A_0 \in H^1$ and slightly improved the local existence. In fact, Struwe only proved the existence of a weak solution is gauge-equivalent to a smooth smooth of the flow for $0 < t < T$.

Proof of Theorem 3

For simplicity, we assume that $x_0 = 0$. For any small constant $\varepsilon > 0$, there is a constant $r_0 > 0$ such that for all $t \in [0, t_1]$ and

$$\int_{B_{r_0}} |F_A(t)|^2 \, dx \le \varepsilon.$$

For simplicity, we assume that $x_0 = 0$. For any small constant $\varepsilon > 0$, there is a constant $r_0 > 0$ such that for all $t \in [0, t_1]$ and

$$\int_{B_{r_0}} |F_A(t)|^2\, dx \le \varepsilon.$$

At $t = 0$, it follows from Uhlenbeck's gauge fixing theorem that there is a smooth gauge transformation $S_0 = S(0)$ and a connection $D_{a(0)} = S_0^*(D_{A(0)}) = d + a(0)$ satisfying

$$d^* a(0) = 0 \quad \text{in } B_{r_0}, \quad a(0)\cdot\nu = 0 \text{ on } \partial B_{r_0}$$

and

$$\int_{B_{r_0}} \frac{|a(0)|^p}{r_0^p} + |\nabla a(0)|^p \le C \int_{B_{r_0}} |F_{a(0)}|^p\, dx$$

for any $p \ge 2$.

Next, we follow the procedure of Uhlenbeck to fix a Coulomb gauge in a neighborhood of $t = 0$.
Using the gauge transformation S_0,

$$\tilde{A}(t) = S_0^{-1} dS_0 + S_0^{-1} A(t) S_0.$$

Next, we follow the procedure of Uhlenbeck to fix a Coulomb gauge in a neighborhood of $t = 0$.
Using the gauge transformation S_0,

$$\tilde{A}(t) = S_0^{-1} dS_0 + S_0^{-1} A(t) S_0.$$

The new connection $D_{\tilde{A}(t)} = S^*(0)(D_{A(t)}) = d + \tilde{A}(t)$ is also a smooth solution the Yang-Mills flow in $\bar{U} \times [0, t_1]$ with $a(0) = \tilde{A}(0)$.

However, $\tilde{A}(t)$ does not satisfy the boundary condition of $A \cdot \nu = 0$ on ∂U, so we cannot apply Lemma 2.7 of Uhlenbeck [**?**] to fix a Columbus gauge for $\tilde{A}(t)$ near $t = 0$.

To sort out this issue of the boundary condition, it follows from Lemma 2.6 of Uhlenbeck [**?**] to get that there are gauge transformations $e^{u_1(t)}$ such that

$$(e^{-u_1(t)})^*(D_{\tilde{A}(t)}) = e^{-u_1(t)} \circ (d + \tilde{A}(t)) \circ e^{u_1(t)} = d + a_1(t)$$

with $a_1(t) := \tilde{A}(0) + \lambda(t)$ and

$$\lambda(t) = -\tilde{A}(0) + e^{-u_1(t)} d e^{u_1(t)} + e^{-u_1(t)} (\tilde{A}(t)) e^{u_1(t)}. \qquad (7)$$

In fact, it can be chosen that

$$u_1(t) = \varphi(\frac{\partial}{\partial r} - \Delta_{S^3})^{-1}(x \cdot (\tilde{A}(t) - \tilde{A}(0))) \qquad (8)$$

with $u_1(t)|_{\partial U} = 0$, where $\varphi(r)$ is a smooth cut-off function in $[0, 1]$ with $\varphi(r) = 1$ near 1 and $\varphi(r) = 0$ near 0.

Lemma 5

For a given function, let v be a solution of the heat equation on $S^3 \times [0,1]$ *satisfying*

$$\partial_r v = \Delta_{S^3} v + f$$

with $v(\theta, 1) = 0$ *on* S^3. *Let* $\varphi(r)$ *be a smooth cut-off function in* $[0,1]$ *with* $\varphi(r) = 1$ *near* 1 *and* $\varphi(r) = 0$ *for* $[0, \delta]$ *with* $\delta > 0$. *Then we have*

$$\|\varphi v\|_{W^{1,p}(S^3 \times [0,1])} \leq C\|f\|_{L^p(S^3 \times [0,1])}$$

and

$$\|\varphi v\|_{W^{2,p}(S^3 \times [0,1])} \leq C\|f\|_{W^{1,p}(S^3 \times [0,1])}$$

for all $p > 1$.

More precisely, $u_1(t) = 0$, $de^{u_1(t)} = du_1(t)$ on ∂U for all $t \in [0, \delta_1]$, which implies $\nu \cdot \lambda(t) = 0$ on ∂U, which implies that the new connection $a_1(t)$ satisfies the required boundary condition $a_1(t) \cdot \nu = 0$ in Lemma 2.7 of Uhlenbeck's paper [**?**].

Moreover, differentiating equation (8) in t yields

$$\frac{\partial u_1(t)}{\partial t} = \varphi\left(\frac{\partial}{\partial r} - \Delta_{S^3}\right)^{-1}\left(x \cdot \frac{\partial \tilde{A}}{\partial t}\right).$$

By applying the L^p-estimate in Lemma 5 again, we have

$$\int_U |\nabla \frac{\partial u_1(t)}{\partial t}|^2 \, dx \leq C \int_U |\frac{\partial A}{\partial t}|^2(\cdot, t) \, dx \leq C \int_U |\nabla F_A|^2(\cdot, t) \, dx$$

for any $t \in [0, \delta_1]$.

By a lemma, we can prove

$$|\nabla s_1(t)| \leq C|\nabla \frac{\partial u_1}{dt}| + C|\nabla u_1|\,|\frac{\partial u_1}{dt}|$$

for all $t \in [0, \delta_1]$ for a sufficiently small $\delta_1 > 0$. By the Sobolev inequality and noticing that $u_1(t) = 0$ on ∂U, we have

$$\begin{aligned} &\int_U |\nabla s_1(t)|^2\, dx \\ \leq\ & C\int_U |\nabla \frac{\partial u_1}{dt}|^2\, dx + (\int_U |\nabla u_1|^4\, dx)^{1/2}(\int_U |\frac{\partial u_1}{dt}|^4\, dx)^{1/2} \\ \leq\ & C\int_U |\nabla \frac{\partial u_1}{dt}|^2\, dx \leq C\int_U |\nabla F_A|^2\, dx. \end{aligned}$$

For any small constant $\varepsilon > 0$, there is a $\delta_1 > 0$ such that for any $t \in [0, \delta_1]$ we have

$$\int_U |\nabla(\tilde{A}(t) - \tilde{A}(0))|^p + |\tilde{A}(t) - \tilde{A}(0)|^p \, dx \leq \varepsilon^p \qquad (9)$$

for $p \in (2, 4]$. By the L^p-estimate, we have

$$\|u_1(t)\|_{W^{2,p}(U)} \leq C\|\tilde{A}(t) - \tilde{A}(0)\|_{W^{1,p}(U)} \leq C\varepsilon$$

for all $t \in [0, \delta_1]$ and hence $|u_1(t)| \leq C\varepsilon$ for all $t \in [0, \delta_1]$. We note that

$$\lambda(t) = e^{-u_1(t)}\tilde{A}(0)e^{u_1(t)} - \tilde{A}(0) + e^{-u_1(t)}de^{u_1(t)}$$
$$+e^{-u_1(t)}(\tilde{A}(t) - \tilde{A}(0))e^{u_1(t)}.$$

Then

$$\|\lambda(t)\|_{W^{1,p}(U)} \leq C\|\tilde{A}(t) - \tilde{A}(0)\|_{W^{1,p}(U)} \leq C\varepsilon.$$

Without loss of generality, we assume that $D_{ref} = d$ and $U = B_{r_0} = B_1$. Our proofs are heavily relied on Lemma 2.7 of Uhlenbeck's paper [**?**]:

Lemma 6

Let $A(0)$ be a connection with $d^*A = 0$ in U with $A \cdot \nu = 0$ on ∂U and satisfy

$$\|A\|_{L^4(U)} \leq k(n)$$

for a small constant $k(n)$. Then there is a small constant $\varepsilon > 0$ such that if $\|\lambda\|_{W^{1,p}(U)} \leq C\varepsilon$ for some $p > 2$ and $\lambda \cdot \nu = 0$, then there is a gauge transformation $S = e^u$ to solve

$$d^*a = d^*(S^{-1}dS + S^{-1}(\tilde{A} + \lambda)S) = 0 \qquad (10)$$

in U with $a \cdot \nu = 0$ on ∂U.

By using Lemma 2.7 of Uhlenbeck [?], there is a small constant $\varepsilon > 0$ such that if $\|\lambda(t)\|_{W^{1,p}(U)} \leq C\varepsilon$, then there is a gauge transformation $S_2(t) = e^{u_2(t)}$ to solve

$$d^* a = d^*(S_2^{-1} dS_2 + S_2^{-1}(\tilde{A}(0) + \lambda(t))S_2) = 0$$

in U with $a \cdot \nu = 0$ on ∂U, where $D_a = S_2^*(D_{a_1})$ satisfies

$$\frac{\partial a}{\partial t} = -D_a^* F_a + D_a s \qquad (11)$$

with $s = S_2^{-1}(t) s_1(t) S_2(t) + S_2^{-1}(t) \circ \frac{dS_2}{dt}$.

In fact, during the existence proof of $u_2(t)$ in Lemma 2.7 of Uhlenbeck [?], it can be chosen that $\nabla u_2 \cdot \nu = 0$ on ∂U and $\int_U u_2(t)\, dx = 0$ for all $t \in [0, \delta_1]$.

In fact, It follows from Lemma 2.7 of [**?**] that we choose the norm $\|\nabla u_2(t)\|_{W^{1,q}(U)}$ for $q > 4$ is bounded since $\|\lambda(t)\|_{W^{1,p}(U)}$ is very small. Since $\int_U u_2(t)\,dx = 0$, we have $\int_U \frac{\partial u_2}{\partial t}\,dx = 0$. Since $\int_U |\nabla u_2|^4$ can be chosen to be small,

$$\int_U |\nabla \frac{\partial u_2}{\partial t}|^2\,dx \leq C \int_U |\nabla s_2(t)|^2 + |\nabla u_2|^2 \, |\frac{\partial u_2}{\partial t}|^2\,dx$$
$$\leq C \int_U |\nabla s_2(t)|^2\,dx + C \left(\int_U |\nabla u_2|^4 \right)^{1/2} \left(\int_U |\frac{\partial u_2}{\partial t}|^4\,dx \right)^{1/2}$$
$$\leq C \int_U |\nabla s_2(t)|^2\,dx + \frac{1}{2} \int_U |\nabla \frac{\partial u_2}{\partial t}|^2\,dx.$$

It implies that

$$\int_U |s_2(t)|^2\,dx \leq C \int_U |\frac{\partial u_2}{\partial t}|^2\,dx \leq \int_U |\nabla \frac{\partial u_2}{\partial t}|^2\,dx \leq C \int_U |\nabla s_2(t)|^2\,dx.$$

Using the fact that $d^*a = 0$ in U and $a \cdot \nu = 0$ on ∂U, it implies from Lemma 2.5 of [?] that for all $t \in [0, \delta_1]$

$$\int_U |a(\cdot,t)|^2 + |\nabla a(\cdot,t)|^2 \leq C \int_U |F_a(\cdot,t)|^2 \leq C\varepsilon.$$

Recalling $s(t) = S_2^{-1}(t)s_1(t)S_2(t) + s_2(t)$, we have

$$\int_U \left\langle ds, \frac{\partial a}{\partial t} \right\rangle = \int_U \left\langle \frac{\partial s}{\partial x_k}, \frac{\partial a_k}{\partial t} \right\rangle = \int_U \left\langle s, \frac{\partial d^* a}{\partial t} \right\rangle + \int_{\partial U} \langle s, \partial_t a \cdot \nu \rangle = 0.$$

Then

$$\begin{aligned} &\int_U \left\langle D_a s, \frac{\partial a}{\partial t} \right\rangle \, dx \\ &= \int_U \left\langle ds, \frac{\partial a}{\partial t} \right\rangle + \left\langle [a, S_2^{-1}(t)s_1(t)S_2(t) + s_2(t)], \frac{\partial a}{\partial t} \right\rangle \, dx \\ &\leq \frac{1}{4} \int_U |\frac{\partial a}{\partial t}|^2 \, dx + C\varepsilon \int_U |\nabla s_1|^2 + |\nabla s_2|^2 \, dx. \end{aligned}$$

Using above, we have

$$\int_U |D_a s - \frac{\partial a}{\partial t}|^2 \, dx \leq C \int_U |\nabla_a F_a|^2 \, dx.$$

Since $s(t) = S_2^{-1}(t) s_1(t) S_2(t) + s_2(t)$, we note

$$|\nabla s_2| \leq |D_a s| + |\nabla s_1| + C|\nabla S_2| \, |s_1| + |a|(|s_1| + |s_2|).$$

Note that $\|\nabla S_2\|_{H^1(U)}$ can be bounded. Then

$$\int_U |\nabla s_2|^2 \leq C \int_U |D_a s|^2 + |\nabla s_1|^2 + |\nabla S_2|^2 \, |s_1|^2 + |a|^2(|s_1|^2 + |s_2|^2)$$

$$\leq C \int_U |D_a s|^2 + |\nabla s_1|^2 + C(\int_U |\nabla S_2|^4)^{1/2}(\int_U |s_1|^4 \, dx)^{1/2}$$

$$+C(\int_U |a|^4)^{1/2}(\int_U (|s_1|^4 + |s_2|^4))^{1/2}$$

$$\leq C \int_U |D_a s|^2 + |\nabla s_1|^2 + C\varepsilon \int_U |\nabla s_2|^2 \, dx.$$

Choosing ε sufficiently small, we obtain

$$\int_U |s|^2 + |D_a s|^2 + |\frac{\partial a}{\partial t}|^2 \, dx \leq C \int_U |\nabla_a F_a|^2 \, dx$$

for any $t \in [0, \delta_1]$.
For the above choices of δ_1, we must assume that $\delta_1 \leq t_1$. If $\delta_1 < t_1$, then we repeat the above the procedure starting at $t = \delta_1$ instead of at $t = 0$; i.e. at $t = \delta_1$, there is a gauge transformation $\tilde{S} = S(\delta_1)$ such that $D_a = d + a = \tilde{S}^*(D_A)$ is a smooth in $\bar{U}$ such that at $t = \delta_1$

$$d^* a = 0, \quad \text{in } U, \quad a \cdot \nu = 0 \text{ on } \partial U.$$

Since $\tilde{S}$ is a fixed smooth transformation, $\tilde{S}^*(D_A)$ is also a smooth solution of Yang-Mills flow. Repeating above procedure, there is the same constant $\delta_1 > 0$ such that we define new smooths $\tilde{u}_1(t)$ and $\tilde{u}_2(t)$ on $[\delta_1, 2\delta_1]$ starting at $t = \delta_1$ with $\tilde{u}_2(\delta_1) = 0$. More precisely, there is a new $\delta_2 > 0$ and gauge transformation $S_1(t) = e^{u_1(t)}$ and $\tilde{S}_2(t) = e^{\tilde{u}_2(t)}$ for any $t \in [\delta_1, 2\delta_1]$, with initial values $u_1(\delta_1) = 0$ and $u_2(\delta_1) = 0$, and the new connection

$$D_{a(t)} = S(t)^*(D_{A(t)}) = (e^{u_2(t)})^* \circ (e^{u_1(t)})^* \circ (\tilde{S}^*(D_{A(t)}))$$

for $t \in [\delta_1, 2\delta_1]$ satisfying the same equation (2.19) (or (2.22)) in $U \times [\delta_1, 2\delta_1]$ with initial values $\tilde{u}_2(\delta_1) = 0$ and $\tilde{u}_2(\delta_1) = 0$. We can continue this procedure to $[0, t_1]$ as required.

Proof of Theorem 7

Let A_0 in $H^{1,2}(M)$ be a weak connection. There is a sequence of $\{A_i(0)\}$, which converges strongly to A_0 in $H^{1,2}(M)$. Let $D_{A_i} = d + A_i$ be a sequence of smooth solutions of the Yang-Mills flow in $M \times [0, t_1]$ with initial values $A_i(0)$. By the local existence theorem with smooth initial data, there is a uniform $T > 0$ such that

$$\sup_{0\le t\le T} \int_U |F_{A_i(t)}|^2 \le \varepsilon.$$

By the above result, there are gauge transformations $S_i(t) = e^{u_i(t)}$ and connections $D_{a_i} = S_i^*(D_{A_i}) = d + a_i$ such that

$$d^* a_i = 0 \quad \text{in } B_{r_0}(x_0), \quad a \cdot \nu = 0 \text{ on } \partial B_{r_0}(x_0),$$

satisfying that all $t \in [0, T)$,

$$\int_U |a_i(t)|^2 + |\nabla a_i(t)|^2 \, dx \le C \int_U |F_{a_i(t)}|^2 \, dx.$$

D_{a_i} is a smooth solution of the equation

$$\frac{\partial a_i}{\partial t} = -D_{a_i}^* F_{a_i} + D_{a_i} s_i \tag{12}$$

in $B_{r_0}(x_0) \times [0, T]$, where

$$s_i(t) = S_i^{-1}(t) \circ \frac{d}{dt} S_i(t).$$

Then we have

$$\int_\delta^T \int_U |D_{a_i} s_i(t)|^2 + |\frac{\partial a_i}{\partial t}|^2 \, dx \, dt \leq C \int_\delta^T \int_U |\nabla_{a_i} F_{a_i}|^2 \, dx \, dt$$

for any $\delta > 0$
As $i \to \infty$ and then $\delta \to 0$, Theorem 7 is proved.

The uniqueness of weak solutions of the Yang-Mills flow

It is known that Struwe [**?**] proved the uniqueness of weak solutions of the Yang-Mills flow under an extra condition that A_0 is irreducible; i.e. for all $s \in \Omega^0(adE)$

$$\|s\|_{L^2(M)} \leq C\|D_{A_0}s\|_{L^2(M)}.$$

It has been an open problem about the uniqueness of the weak solution of the Yang-Mills flow in four manifolds with initial data in H^1. We would like to point out that the weak solution constructed by Struwe in [**?**] is a weak limit of smooth solutions.
In this sense, we solve the problem to prove:

Theorem 7

*The weak solution of the Yang-Mills flow (**??**) with initial value A_0 in H^1 is unique.*

For the proof of Theorem 7, we need a kind of parabolic gauge fixing for the Yang-Mills flow. However, in Theorem 2, $d^*a = 0$ in U with Nuemann boundary condition $a \cdot \nu = 0$ on ∂U is not unique. To overcome this difficulty, we improve a key lemma of Uhlenbeck (Lemma 2.7 of [**?**]) from the Neumann boundary condition to the Dirichlet boundary condition. By a covering of M, we glue local connections to a global connection on the whole manifold M to prove the uniqueness of weak solutions of the Yang-Mills flow.

Thank you very much!!!