Schematic Harder-Narasimhan stratification

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Joint work with Sudarshan Gurjar.

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- This gives an open substack $Coh_{X/S}^{pure}$ of the algebraic stack $Coh_{X/S}$.

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- Note that a non-zero, coherent, pure E is semistable if and only if $\ell = 1$, and its HN-filtration is $0 \subset E$.
- The sequence $\mathrm{HN}(E) = (p(E_1, \lambda), \ldots, p(E_\ell, \lambda))$ in $\mathbb{Q}[\lambda]$ is called the **Harder-Narasimhan type** of E. There is a natural **partial order** on the set of all such sequences in $\mathbb{Q}[\lambda]$. Semistable type is the minimum for any given $p(E, \lambda)$.

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- a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_\ell = E_T$ such that each E_i/E_{i-1} is flat over T and for each $t \in T$, the restriction $0 = E_0|X_t \subset E_1|X_t \subset \ldots \subset E_\ell|X_t = E|X_t$ is the HN-filtration of E_t with HN-type τ.

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- **Theorem** (arXiv 0909.0891) : There exists a unique structure of a locally closed subscheme $S^{\tau}(E) \subset S$ on $|S|^{\tau}(E) \subset |S|$, such that for any S-scheme T, the morphism $T \to S$ factors via $S^{\tau}(E) \subset S$ if and only if there exists a relative HN-filtration of $E/X/S$ over T, with constant type τ. Moreover, a relative HN-filtration over T, if it exists, is unique.

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- Given an HN-type τ, let $Coh^\tau_{X/S}(T) \subset Coh^{pure}_{X/S}(T)$ be the full sub-groupoid consisting of all $\mathcal{F}$ on X_T which admit a relative HN-filtration over T. Clearly, this defines a substack $Coh^\tau_{X/S} \subset Coh^{pure}_{X/S}$. As a consequence of the above theorem, $Coh^\tau_{X/S}$ is an algebraic stack which is a locally closed substack of $Coh^{pure}_{X/S}$.

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- The above result easily extends (2012, with S.G.) from coherent $\mathcal{O}$-modules to $\mathcal{O}$-coherent Λ-modules, where Λ is a sheaf of rings of split almost-polynomial differential operators on X/S as introduced by Simpson in 1994. Special cases: Higgs bundles, integrable connections, integrable logarithmic connections, etc.

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We fix the following data and notation for the rest of this talk:

- k a base field of characteristic zero.
- G a reductive group scheme over k.
- $T \subset B \subset G$ a maximal torus and a Borel. We assume that T is split over k.
- $\Delta \subset X^*(T)$ corresponding set of all simple roots.
- $\omega_\alpha \in \mathbb{Q} \otimes X^*(T)$ dominant weight corresponding to $\alpha \in \Delta$.
- $I_P \subset \Delta$ the set of inverted simple roots corresponding to any standard parabolic $B \subset P \subset G$.
- $\overline{C} \subset \mathbb{Q} \otimes X_*(T)$ closed positive Weyl chamber. Recall that $\overline{C} = \{\mu \in \mathbb{Q} \otimes X_*(T) \,|\, \langle \alpha, \mu \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$.
- Partial order on $\overline{C}$ defined by putting $\mu \leq \nu$ if $\langle \omega_\alpha, \mu \rangle \leq \langle \omega_\alpha, \nu \rangle$ for all $\alpha \in \Delta$ and $\chi(\mu) = \chi(\nu)$ for all $\chi \in \widehat{G} = Hom(G, \mathbb{G}_m)$.

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- E a principal $G_K = G \otimes_k K$-bundle on X. Parabolics in G_K that contain B_K are exactly the P_K where $B \subset P \subset G$. For simplicity of notation, we write G_K, P_K, B_K, T_K etc simply as G, P, B, T etc. Note that $\widehat{G_K} = \widehat{G}$, $\widehat{P_K} = \widehat{P}$, etc. as T is already split over k.

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- A rational reduction of E is a pair (P, σ) where P is a standard parabolic and σ is a section of $E/P \to X$ over an open subscheme U, such that U is the unique maximal open subscheme over which σ can be prolonged. By valuative criterion for properness, U is **big**, that is, $\dim(X - U) \leq \dim(X) - 2$.

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- E is called **semistable** if for each rational P-reduction (U, σ) and character $\chi \in \widehat{P}$, we have $\deg(\chi_* \sigma^* E) \leq 0$. Here, $\sigma^* E$ is a principal P-bundle on U, $\chi_* \sigma^* E$ is its associated line bundle, and its degree is defined using $\mathcal{O}_X(1)$. This makes sense as U is big.

The type $\mu_{(P,\sigma)}(E)$ of a rational reduction of E

We have natural decompositions

$$\begin{aligned} \mathbb{Q}\otimes X^*(T) &= (\mathbb{Q}\otimes \widehat{P}|_T)\oplus(\oplus_{\alpha\in I_P}\mathbb{Q}\alpha), \text{ where, in turn,}\\ \mathbb{Q}\otimes \widehat{P}|_T &= (\oplus_{\alpha\in\Delta-I_P}\mathbb{Q}\omega_\alpha)\oplus(\mathbb{Q}\otimes\widehat{G}|_T). \end{aligned}$$

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Hence it makes sense to define the type of (P,σ) as the unique element $\mu_{(P,\sigma)}(E)\in\mathbb{Q}\otimes X_*(T)$ such that

$$\langle\chi,\mu_{(P,\sigma)}(E)\rangle = \begin{cases}\deg(\chi_*\sigma^*E) & \text{if } \chi\in\widehat{P}|_T,\\ 0 & \text{if } \chi\in I_P.\end{cases}$$

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- If $\rho : P \to L$ is the Levi quotient of P (by its unipotent radical) then the principal L-bundle $\rho_* \sigma^* E$ is a semistable principal L-bundle defined on the big open subscheme U (note that the definition of semistability makes sense also for a principal bundle defined on any big open subscheme of X).

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- For any non-trivial character $\chi \in \widehat{P}$ whose restriction to the chosen maximal torus $T \subset B \subset P$ has the form $\sum n_i \alpha_i$ with $\alpha_i \in \Delta$, where $n_i \geq 0$, and at least one $n_i \neq 0$, we have $\deg(\chi_* \sigma^* E) > 0$.

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- We have $\mathrm{HN}(E) \in \overline{C} \subset \mathbb{Q} \otimes X_*(T)$. In fact, $\langle \alpha, \mathrm{HN}(E) \rangle > 0$ for all $\alpha \in \Delta - I_P$ and $\langle \beta, \mathrm{HN}(E) \rangle = 0$ for all $\beta \in I_P$, so it lies in the face of $\overline{C}$ corr. to P. The open face corr. to $P = B$, and the vertex $0 \in \overline{C}$ corr. to $P = G$, which is case when E is semistable.

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- **Maximality** Given any other rational parabolic reduction (Q, τ) of E, we have
$$\mu_{(Q,\tau)} \leq \mathrm{HN}(E)$$
w.r.t. the natural partial ordering on $\overline{C}$.

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- A morphism $T \to S$ should factor via $S^\tau(E) \hookrightarrow S$ if and only if the pullback E_T over X_T admits a '**relative canonical reduction**' of constant type τ. Also, a rel. can. red., if it exists, should be unique.

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- **Question** How does one define a relative canonical reduction?

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- **Theorem** (arXiv 1208.5572) : S has a stratification by locally closed subschemes $S^\tau(E)$ indexed by $(\overline{C}, \leq)$, such that a base change E_T admits a relative canonical reduction of constant HN-type τ if and only if $T \to S$ factors via $S^\tau(E) \hookrightarrow S$. Moreover, a relative canonical reduction over T, when it exists, is unique.

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- For any standard parabolic P, the anti-canonical bundle $\omega_{G/P}^{-1}$ is very ample on G/P.
- We get an embedding $G/P \hookrightarrow \mathbf{P}(V_P)$ where V_P is the G-rep $H^0(G/P, \omega_{G/P}^{-1})^\vee$.

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Preliminaries in higher dimensions

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- Given E over S, we get an associated vector bundle $E(V_\lambda)$ over X, and a closed imbedding $E/P \hookrightarrow \mathbf{P}(E(V_P))$.
- If $\sigma : U \to E_s/P$ is a rational P-reduction of $E_s = E|X_s$ for some $s \in S$, then we get a line subbundle $L' \subset E(V_\lambda)|U$ over U. This uniquely up to unique isomorphism extends to a line bundle L on X_s with a homomorphism $L \to E(V_\lambda)$. This inspires our definition of a relative rational reduction.

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- We require that L' admits a prolongation to a line bundle on X.
- If $T = \mathrm{Spec}\, K$ for any field K over S, then note that the above definition reduces to the usual definition over X_K. Also, it reduces to the earlier definition when X/S is of relative dimension 1.

Prolonging line bundles from big open subschemes

- **Question.** Let $X \to S$ be smooth, projective. Let $U \subset X$ be relatively big open subscheme over S. Then does every line bundle L' on U admit a prolongation to X?
- This is certainly true for $S = \mathrm{Spec}\, k$ for any field k. For L' will be defined by a Weil divisor D on U, and we can define $L = \mathcal{O}_X(\overline{D})$ where $\overline{D}$ is the closure of D in X.
- While framing the definition of a rel rational P-reduction, we expected that in general the above question has a negative answer, even in relative dimension 2. This is indeed so, as shown by the following.
- **Example** (Najmuddin). Let $S = \mathrm{Spec}\, k[\epsilon]/(\epsilon^2)$. Let $X = \mathbf{P}^2_S$. Let $U = X - \{x\} \subset X$ be the complement of any k-valued closed point $x \in \mathbf{P}^2_k$. As $H^1(\mathbf{P}^2_k - \{x\}, \mathcal{O}) \neq 0$, the trivial line bundle $\mathcal{O}_{\mathbf{P}^2_k - \{x\}}$ admits a nontrivial infinitesimal deformation parameterized by S, which defines a nontrivial line bundle L' on U. But as $H^1(\mathbf{P}^2_k, \mathcal{O}) = 0$, L' does not prolong to X.

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Lemma Let $\pi : X \to S$ be a smooth morphism where S is noetherian, let $j : U \hookrightarrow X$ be an open subscheme which is relatively big over S, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module. Then the homomorphism $\mathcal{E} \to j_*(\mathcal{E}|U)$ is an isomorphism.

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Proof. For any $z \in Z = X - U$, if $\pi(z) = s \in S$, then $depth(\mathcal{O}_{X_s,z}) \geq 2$ as $\mathcal{O}_{X_s,z}$ is a regular local ring of dimension ≥ 2. By EGA IV_2 Proposition 6.3.1, we have $depth(\mathcal{O}_{X,z}) = depth(\mathcal{O}_{S,s}) + depth(\mathcal{O}_{X_s,z})$, hence $depth(\mathcal{O}_{X,z}) \geq 2$. Therefore, $depth_Z(\mathcal{O}_X) = \inf_{z\in Z} depth(\mathcal{O}_{X,z}) \geq 2$. Hence the desired conclusion follows from EGA IV_2 Theorem 5.10.5. $\square$

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- **2.** The section $\sigma : U \to \mathbf{P}(E(V_P))$ defined by the line subbundle $L|U \subset E(V_P)|U$ factors via the closed subscheme $(E/P)|U \subset E(V_P)|U$.

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- **Corollary** If S is reduced and $\mathrm{HN}(E_s)$ is constant over S, then there exists a relative canonical reduction over S. In particular, there exists a relatively big open $U \subset X$ over S, and a parabolic reduction over U which restricts to the canonical reduction on each fiber.

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- Note that the statement above **does not** refer to our new definition of a relative rational reduction.
- The result is new even for $G = GL_n$, as it refers to μ-semistability, not Gieseker semistability.

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- **Corollary** For any $\tau \in \overline{C}$, let $Bun^{\tau}_{X/S}(G)$ be the S-groupoid which associates to any $T \to S$ the category whose objects principal G-bundles E on X_T together with a relative canonical reduction (U, σ) of type τ, and whose morphisms are isomorphisms of principal bundles. Then $Bun^{\tau}_{X/S}(G)$ is an algebraic stack which is a locally closed substack of the stack $Bun_{X/S}(G)$ of all principal G-bundles on X/S.

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- Proof: An easy descent argument shows that $Bun^{\tau}_{X/S}(G)$ is a stack. The Main Theorem shows that the forgetful 1-morphism $Bun^{\tau}_{X/S}(G) \to Bun_{X/S}(G)$ is relatively representable by a locally closed embedding.

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- Under this hypothesis, $Bun^{\tau}_{X/S}(G)$ can be shown to be algebraic stacks for k of any characteristic (see arXiv:1605.08997). However, they may not embed into $Bun_{X/S}(G)$.
- The proof of the above uses the universal property of the projectivization of Grothendieck's Q-sheaf.

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- The proof of the Main Theorem in higher dimensions uses standard techniques from projective geometry (Grothendieck complex, flattening stratifications, Castelnuovo-Mumford regularity, Quot schemes, existence and properties of relative Picard schemes, relative duality, cohomology vanishing result of the Enriques-Severi-Zariski kind, etc).

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- The proof is by induction on the rel. dim. of X/S. For the inductive step we proved the following result, an ingredient of which is an analog for principal bundles of the semistable restriction theorem of Mehta-Ramanathan for μ-semistable torsion-free sheaves.

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- **Proposition** Let $(X, \mathcal{O}_X(1))$ be a smooth connected projective variety over an extension field K of k, of dimension ≥ 2. Given E on X, there exists $m_0 \geq 1$ s.t. for any general smooth hypersurface $Y \subset X$ where $Y \in |\mathcal{O}_X(m)|$ where $m \geq m_0$, we have

$$m \cdot \mathrm{HN}(E) = \mathrm{HN}(E|Y).$$

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  Moreover, if $\sigma : U \to E/P$ is the canonical reduction of E then $U \cap Y$ is big in Y, and $\sigma|(U \cap Y)$ is the canonical reduction of $E|Y$.
- For an arbitrary smooth $Y \in |\mathcal{O}_X(m)|$ (not necessarily 'general'), we have $m \cdot \mathrm{HN}(E) \leq \mathrm{HN}(E|Y)$.

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- With all these ingredients, the method of the proof of semicontinuity of $\mathrm{HN}(E_s)$, of uniqueness of a rel. can. red. whenever it exists, and of the Main Theorem, involves taking any point $s \in S$, finding a 'good' hypersurface H for restricting E_s to it, then spreading H to a smooth relative hypersurface Y in a neighbourhood W of $s \in S$.

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- Note that even if $Y_s = H \subset X_s$ is general for E_s, we cannot assume that $Y_t \subset X_t$ is general for E_t for all t in some sufficiently small neighbourhood of s in S.
- The argument proceeds by comparing the canonical reductions of E_t and $E_t|Y_t$ for $t \in W$, comparing the relative Picard schemes of X_W/W and Y/W, and then trying to lift the relative reduction over the stratification for $E|Y$ to E over X by 'shrinking' the strata as little as possible.

Thank you for your kind attention!