Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces

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Introduction

Harmonic bundles

Let X be a Riemann surface. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on X, i.e., $(E, \overline{\partial}_E)$ is a holomorphic vector bundle on X with $\theta \in H^0(X, \mathrm{End}(E) \otimes \Omega_X^1)$.
Let h be a Hermitian metric of E.

- **the Chern connection ∇_h determined by $\overline{\partial}_E$ and h.**
- **the adjoint $\theta_h^\dagger$ of θ with respect to h.**

Definition h **is called** *harmonic metric* **of** $(E, \overline{\partial}_E, \theta)$ **if**

$$\nabla_h \circ \nabla_h + [\theta, \theta_h^\dagger] = 0 \quad \textit{(Hitchin equation)}.$$

$$(\Longleftrightarrow \textbf{the connection } \mathbb{D}_h^1 := \nabla_h + \theta + \theta_h^\dagger \textbf{ is flat})$$

$(E, \overline{\partial}_E, \theta, h)$ **is called** *harmonic bundle*.

Basic examples

Rank 1 case **If** $\operatorname{rank} E = 1$,

$$\begin{array}{c}(E, \overline{\partial}_E, \theta, h) \\ \textbf{harmonic bundle}\end{array} \iff \begin{cases} \theta \in H^0(X, \Omega_X^1) \\ h \text{ is flat, i.e., } \nabla_h \circ \nabla_h = 0 \end{cases}$$

Complex variation of Hodge structure

A harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ **is called** *complex variation of Hodge structure* **if**

- $E = \bigoplus E^i$ **(orthogonal decomposition)**
- $\theta(E^i) \subset E^{i-1} \otimes \Omega_X^1$

Example

When $f : Y \longrightarrow X$ **is a smooth projective fibration, the cohomology groups** $H^j(f^{-1}(x), \mathbb{C})$ $(x \in X)$ **gives a flat bundle on** X**, which is naturally a complex variation of Hodge structure.**

Fundamental theorem

A harmonic bundle has the underlying Higgs bundle and the underlying flat bundle

$$(E, \overline{\partial}_E, \theta) \longleftarrow (E, \overline{\partial}_E, \theta, h) \longrightarrow (E, \mathbb{D}^1)$$

Theorem (Corlette-Donaldson-Hitchin-Simpson)

When X is compact, we have the following homeomorphisms

$$\begin{pmatrix} \textbf{moduli of} \\ \textbf{Higgs bundles} \\ \textbf{(polystable, } \deg = 0\textbf{)} \end{pmatrix} \xleftarrow{\simeq} \begin{pmatrix} \textbf{moduli of} \\ \textbf{harmonic bundles} \end{pmatrix} \xrightarrow{\simeq} \begin{pmatrix} \textbf{moduli of} \\ \textbf{flat bundles} \\ \textbf{(semisimple)} \end{pmatrix}$$

The induced family parametrized by $\mathbb{C}^*$

Suppose X is compact.

$$\begin{pmatrix} \text{moduli of} \\ \text{Higgs bundles} \end{pmatrix} \xleftarrow{\simeq} \begin{pmatrix} \text{moduli of} \\ \text{harmonic bundles} \end{pmatrix} \xrightarrow{\simeq} \begin{pmatrix} \text{moduli of} \\ \text{flat bundles} \end{pmatrix}$$

Let $(E, \overline{\partial}_E, \theta, h)$ is a harmonic bundle on X.
We have the obvious deformation of Higgs bundle $(E, \overline{\partial}_E, t\theta)$ $(t \in \mathbb{C}^*)$.

$$\begin{array}{ccccc} \text{Higgs bundles} & & \text{harmonic bundles} & & \text{flat bundles} \\ (E, \overline{\partial}_E, t\theta) & \longleftrightarrow & (E, \overline{\partial}_E, t\theta, h_t) & \longleftrightarrow & (E, \nabla_{h_t} + t\theta + \bar{t}\theta^{\dagger}_{h_t}) \\ (t \in \mathbb{C}^*) & & (t \in \mathbb{C}^*) & & (t \in \mathbb{C}^*) \end{array}$$

This gives a $\mathbb{C}^*$-action on the moduli spaces.

Easy examples

Rank 1 case **If** $\operatorname{rank} E = 1$**,**

$$\begin{array}{ccccc} \textbf{Higgs bundles} & & \textbf{harmonic bundles} & & \textbf{flat bundles} \\ (E, \overline{\partial}_E, t\theta) & \longleftrightarrow & (E, \overline{\partial}_E, t\theta, h) & \longleftrightarrow & (E, \nabla_h + 2\operatorname{Re}(t\theta)) \end{array}$$

Complex variation of Hodge structure (CVHS)

If $(E, \overline{\partial}_E, \theta, h)$ **is CVHS,** $(E, \overline{\partial}_E, t\theta) \simeq (E, \overline{\partial}_E, \theta)$ **for any** $t \neq 0$**.**

$$\begin{array}{ccccc} \textbf{Higgs bundles} & & \textbf{harmonic bundles} & & \textbf{flat bundles} \\ (E, \overline{\partial}_E, t\theta) & \longleftrightarrow & (E, \overline{\partial}_E, \theta, h) & \longleftrightarrow & (E, \nabla_h + \theta + \theta_h^{\dagger}) \end{array}$$

(CVHS $\Longleftrightarrow$ **fixed point of the moduli)**

If $(E, \overline{\partial}_E, \theta, h) = \bigoplus (L_i, \overline{\partial}_{L_i}, \varphi_i, h_{L_i}) \otimes (E_i, \overline{\partial}_{E_i}, \theta_i, h_{E_i})$**, (**$\operatorname{rank} L_i = 1$**,** $(E_i, \overline{\partial}_{E_i}, \theta_i, h_{E_i})$ **CVHS), then**

$$(E, \overline{\partial}_E, t\theta, h_t) \simeq \bigoplus (L_i, \overline{\partial}_{L_i}, t\varphi_i, h_{L_i}) \otimes (E_i, \overline{\partial}_{E_i}, \theta_i, h_{E_i})$$

$$(E, \mathbb{D}^1_{h_t}) \simeq \bigoplus (L_i, \nabla_{L_i} + 2\operatorname{Re}(t\varphi_i)) \otimes (E_i, \mathbb{D}^1_{h_{E_i}})$$

General issue

In general, it is very difficult to describe the families $(E,\overline{\partial}_E,t\theta,h_t)$ and $\mathbb{D}^1_{h_t}$.

We are interested in the behaviour of $(E,\overline{\partial}_E,t\theta,h_t)$ $(t\to\infty)$.

The behaviour $(E,\overline{\partial}_E,t\theta,h_t)$ $(t\to 0)$ was studied by *Hitchin* and *Simpson*.

$$(E,\overline{\partial}_E,t\theta,h_t) \xrightarrow{t\to 0} \quad \textit{complex variation of Hodge structure}$$
$$\textit{(fixed point of the moduli)}$$

More recently, the behaviour $(E,\overline{\partial}_E,t\theta,h_t)$ $(t\to\infty)$ has been studied by several groups of mathematicians from several viewpoints.

It is motivated by the interest to the asymptotic of various structures of the moduli spaces around infinity. For instance, it is interesting and challenging to study the asymptotic of the homeomorphism

$$\begin{pmatrix} \text{moduli of} \\ \text{Higgs bundles} \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} \text{moduli of} \\ \text{flat bundles} \end{pmatrix}$$

It might be useful to see how the family $(E,\overline{\partial}_E,t\theta)$ is transformed by Φ.

Mazzeo-Swoboda-Weiss-Witt

The important contributions were given by *Mazzeo, Swoboda, Weiss and Witt*.

Asymptotic decoupling

$$\nabla_h \circ \nabla_h + [\theta, \theta_h^\dagger] = 0 \quad \textit{(Hitchin equation)}$$
$$\nabla_h \circ \nabla_h = 0,\ [\theta, \theta_h^\dagger] = 0 \quad \textit{(decoupled Hitchin equation)}$$

decoupled Hitchin equation $\Longrightarrow (E, \overline{\partial}_E, \theta, h) \overset{\text{loc}}{=} \bigoplus (L_i, \overline{\partial}_{L_i}, \varphi_i, h_{L_i}) \quad (\operatorname{rank} L_i = 1)$.

When t **is large,** $\nabla_{h_t} \circ \nabla_{h_t}$ **and** $[t\theta, \bar{t}\theta_{h_t}^\dagger]$ *should* **be close to** 0 **on** $X \setminus D(E, \theta)$ **(under some assumption), i.e.,** $(E, \overline{\partial}_E, t\theta, h_t) \overset{\text{loc}}{\sim} \bigoplus (L_i, \overline{\partial}_{L_i}, t\theta_i, h_{L_i,t})$ **on** $X \setminus D(E, \theta)$

Limiting configuration

$\lim_{t\to\infty} h_{L_i}$ *should* **exist (after gauge transformations).**

It is desirable to have explicit descriptions of the limiting configuration.

They established the case where $\operatorname{rank} E = 2$ **and the spectral curve of** $(E, \overline{\partial}_E, \theta)$ **is smooth.**

Katzarkov-Noll-Pandit-Simpson

A completely different viewpoint was given by *Katzarkov, Noll, Pandit and Simpson*. **They are interested in the asymptotic behaviour of the parallel transports of** $\mathbb{D}^1_{h_t}$

$$\Pi_{\gamma,t} : E_{\gamma(0)} \longrightarrow E_{\gamma(1)} \quad (t \to \infty)$$

along a path $\gamma : [0,1] \longrightarrow X$ *(Hitchin-WKB problem)*.

They proposed a conjectural estimate on how $\Pi_{\gamma,t}$ **are far from unitary (under some assumption on** γ **and** θ**).**

Let $\widetilde{X} \longrightarrow X$ **be a universal covering. The family of harmonic bundles** $(E, \overline{\partial}_E, t\theta, h_t)$ **induces a family of harmonic maps**

$$\varphi_t : \widetilde{X} \longrightarrow \mathscr{H}_r := \mathrm{GL}(r,\mathbb{C})/U(r) \quad (r = \operatorname{rank} E)$$

Katzarkov-Pandit-Noll-Simpson **proved that the estimate implies that any sequence** φ_{t_i} **contains a subsequence which converges to a harmonic map from** $\widetilde{X}$ **to the affine building of** A_{r-1}**-type.**

Collier-Li

Collier and Li **closely studied the asymptotic of some cyclic type harmonic bundles.**

- **They established the asymptotic decoupling and the convergence to the limiting configuration. They also described the limit metric precisely.**
- **They applied their result to the Hitchin-WKB problem.**

Our results

Asymptotic decoupling

- **OK in any rank case.**

  We do not need the compactness of X, the smoothness and the irreducibility of the spectral curve.

  We need the eigenvalues of the Higgs field are generically multiplicity-free.

- **We can apply it to the Hitchin-WKB problem, and obtain the conjectural estimate.**

Limiting configuration

- **OK in the rank 2 case**
- **We have a rather explicit description of the limiting configuration.**

We shall explain our result in the case $\mathrm{rank}\, E = 2$.

Asymptotic decoupling

Preliminary

Let $(E, \overline{\partial}_E, \theta, h)$ **be a harmonic bundle of rank** 2 **on a compact Riemann surface** X**.**

$$\Sigma(E,\theta) = \textbf{the spectral curve of } (E,\theta) \subset T^*X$$

$$\Sigma(E,\theta)_P = \Sigma(E,\theta) \times_X \{P\} = \{\textbf{eigenvalue of } \theta \textbf{ at } P\}$$

We have two cases. (Note $\#\Sigma(E,\theta)_P \leq 2$ **if** $\mathrm{rank}\, E = 2$**.)**

(i) $\#\Sigma(E,\theta)_P = 1$ **for any** $P \in X$

(ii) **We have a finite subset** $D(E,\theta) \subset X$ **(the discriminant of** (E,θ)**), and** $\#\Sigma(E,\theta)_P = 2$ **for any** $P \in X \setminus D(E,\theta)$**.**

(i) $\#\Sigma(E,\theta)_P = 1$ *for any* $P \in X$

The case (i) is easy.

$$(E,\overline{\partial}_E,\theta,h) = (L,\overline{\partial}_L,\varphi,h_L)\otimes(E',\overline{\partial}_{E'},\theta',h') \quad (\mathrm{rank}\,L = 1,\ \theta' \textbf{ nilpotent}).$$

$$(E,\overline{\partial}_E,t\theta,h_t) = (L,\overline{\partial}_L,t\varphi,h_L)\otimes(E',\overline{\partial}_{E'},t\theta',h'_t)$$

Because θ' **is nilpotent, the family** $(E',\overline{\partial}_{E'},t\theta',h_{t'})$ **is convergent when** $t\to\infty$**:**

$$\lim_{t\to\infty}(E',\overline{\partial}_{E'},t\theta',h_{t'}) = \textbf{complex variation of Hodge structure}$$

Hence,

$$(E,\overline{\partial}_E,t\theta,h_t) \sim (L,\overline{\partial}_L,t\varphi,h_L)\otimes \textbf{(a CVHS)}$$

$$(E,\nabla_t + t\theta + t\theta_h^{\dagger}) \sim (L,\nabla_{h_L} + 2\mathrm{Re}(t\varphi))\otimes \textbf{(a flat bundle)}$$

We are more interested in the case *(ii)* $\#\Sigma(E,\theta)_P = 2\ (\forall P \in X \setminus D(E,\theta))$

We have a ramified covering $p : X' \longrightarrow X$ such that

$$\Sigma\big(p^*(E,\theta)\big) = \mathrm{Im}(\phi_1) \cup \mathrm{Im}(\phi_2), \quad (\phi_1, \phi_2 \in H^0(X', \Omega^1_{X'}),\ \phi_1 \neq \phi_2)$$

$$\left(\begin{array}{c}\text{the asymptotic behaviour of}\\ (E, \overline{\partial}_E, t\theta, h_t)\ (t \to \infty)\end{array}\right) \longleftrightarrow \left(\begin{array}{c}\text{the asymptotic behaviour of}\\ p^*(E, \overline{\partial}_E, t\theta, h_t)\ (t \to \infty)\end{array}\right)$$

Assumption (inessential)

We may assume $\Sigma(E,\theta) = \mathrm{Im}(\phi_1) \cup \mathrm{Im}(\phi_2)$ **(**$\phi_1, \phi_2 \in H^0(X, \Omega^1_X)$**,** $\phi_1 \neq \phi_2$**)**

We have $D(E,\theta) = \{$zero of $\phi_1 - \phi_2\}$ and

$$(E, \overline{\partial}_E, \theta)_{|X \setminus D(E,\theta)} = \bigoplus_{i=1,2} (E_i, \overline{\partial}_{E_i}, \phi_i\, \mathrm{id}_{E_i})$$

Asymptotic decoupling

Theorem

Fix a Kähler metric g_X of X. For any neighbourhood N of $D(E,\theta)$, we have positive constants C_1, ε_1 such that the following holds on $X \setminus N$.

- **Let v_i be local sections of E_i $(i = 1,2)$.**

$$|h_t(v_1, v_2)| \leq C_1 \exp(-\varepsilon_1 |t|) |v_1|_{h_t} \cdot |v_2|_{h_t} \quad (\textit{Asymptotic orthogonality})$$

$$\Big(\Longleftrightarrow \big|[t\theta, \bar{t}\theta^\dagger_{h_t}]\big|_{h_t, g_X} = \big|\nabla_{h_t} \circ \nabla_{h_t}\big|_{h_t, g_X} = O\big(\exp(-\varepsilon_1 |t|)\big) \Big)$$

  We also have the estimate of the derivatives.

- **Let h_{t,E_i} denote the restriction of h_t to E_i.**

$$\big|\nabla_{h_{t,E_i}} \circ \nabla_{h_{t,E_i}}\big|_{h_t, g_X} \leq C_1 \exp(-\varepsilon_1 |t|)$$

$(E, \overline{\partial}_E, t\theta, h_t)$ $(t >> 0)$ **is close to a direct sum of rank one harmonic bundles.**

(OK in the higher rank case if (E,θ) is generically regular semisimple.)

Mazzeo, Swoboda, Weiss, Witt: **the case** $\mathrm{rank}\, E = 2$ **and** $\Sigma(E,\theta)$ **smooth**

Collier-Li: **some cyclic type harmonic bundles**

Simpson's main estimate

Let $\Delta(R) := \{z \in \mathbb{C} \mid |z| < R\}$**. Let** $(E, \overline{\partial}_E, \theta, h)$ **be a harmonic bundle on** $\Delta(R)$**.**

We have $\theta = f\,dz \ (f \in \mathrm{End}(E))$**.**

- $|f|_h \leq C_2$ **on** $\Delta(R_1) \ (R_1 < R)$**, where**
  C_2 **depends on** R**,** R_1**,** $\mathrm{rank}\, E$ **the eigenvalues of** f**.**

Suppose $(E, \overline{\partial}_E, \theta) = \bigoplus (E_i, \overline{\partial}_{E_i}, \theta_i)$**. Let** π_i **denote the projection of** E **onto** E_i**.**

- $|\pi_i|_h^2 - \mathrm{rank}\, E_i \leq \exp(-C_3)$ **on** $\Delta(R_2) \ (R_2 < R_1)$**, where**
  C_3 **depends on** R**,** R_1**,** R_2**,** $\mathrm{rank}\, E$ **and the eigenvalues of** f**.**
  When the difference of the eigenvalues are large, C_3 **are large.**

Application to Hitchin-WKB problem

We have the flat connections $\mathbb{D}^1_{h_t}$ associated to $(E, \overline{\partial}_E, t\theta, h_t)$.
For any C^∞-path $\gamma : [0,1] \longrightarrow X$, we have the parallel transport of $\mathbb{D}^1_{h_t}$:

$$\Pi_{\gamma,t} : E_{|\gamma(0)} \longrightarrow E_{|\gamma(1)}$$

Katzarkov-Noll-Pandit-Simpson asked how $\Pi_{\gamma,t}$ are far from unitary.

"vector distance" of metrics

Let V be an r-dimensional vector space over $\mathbb{C}$.
Let h_1 and h_2 be two Hermitian metrics of V.
$\exists\ e_1, \dots, e_r$ basis of V, orthogonal with respect to both h_1 and h_2.
We set

$$\kappa_j := \log |e_j|_{h_2} - \log |e_j|_{h_1}.$$

We impose $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_r$.

(vector distance) **We set $\vec{d}(h_1, h_2) := (\kappa_1, \dots, \kappa_r) \in \mathbb{R}^r$.**

KNPS proposed an conjectural estimate for $\vec{d}(h_{t|E_{|\gamma(0)}}, \Pi^*_{\gamma,t} h_{t|E_{\gamma(1)}})$ $(t \to \infty)$.

We have $(E,\overline{\partial}_E,\theta)_{|X\setminus D(E,\theta)} = \bigoplus_{i=1,2}(E_i,\overline{\partial}_{E_i},\phi_i\,\mathrm{id}_{E_i})$**.**

$\gamma:[0,1]\longrightarrow X\setminus D(E,\theta)$ *non-critical* $\overset{\text{def}}{\Longleftrightarrow}$ $\gamma^*\,\mathrm{Re}(\phi_1-\phi_2)$ **is nowhere vanishing on** $[0,1]$**.**

We may assume $-\int_\gamma \mathrm{Re}(\phi_1) > -\int_\gamma \mathrm{Re}(\phi_2)$**.**

Theorem

Suppose γ **is non-critical. Then,** $\exists C_{10},\varepsilon_{10}>0$ **depending only on** X**,** γ**,** ϕ_1,ϕ_2 **such that**

$$\left|\frac{1}{t}\vec{d}\big(h_{t|\gamma(0)},\Pi^*_{t,\gamma}h_{t|\gamma(1)}\big)-\Big(-2\int_\gamma \mathrm{Re}(\phi_1),-2\int_\gamma \mathrm{Re}(\phi_2)\Big)\right|\leq C_{10}\exp(-\varepsilon_{10}t)$$

(OK in the higher rank case if (E,θ) **is generically regular semisimple.)**

Katzarkov-Noll-Pandit-Simpson: conjectured, Collier-Li: verified some interesting cases

asymptotic decoupling, singular perturbation theory $\Longrightarrow$ **this estimate**

We have the harmonic maps φ_t $(t>0)$ **from the universal covering** $\widetilde{X}$ **to** $\mathrm{GL}(2)/U(2)$**. Any sequence** φ_{t_i} **contains a subsequence which is convergent to a harmonic map from** $\widetilde{X}$ **to the affine building of** A_1**-type. (KNPS+our estimate)**

Limiting configurations

Recall $(E, \overline{\partial}_E, \theta, h)$ **is a harmonic bundle of rank** 2 **on a compact Riemann surface** X**.**

$$(E, \overline{\partial}_E, \theta)_{|X \setminus D(E,\theta)} = \bigoplus_{i=1,2} (E_i, \overline{\partial}_{E_i}, \phi_i \, \mathrm{id}_{E_i})$$

We have the family $(E, \overline{\partial}_E, t\theta, h_t)$ $(t \in \mathbb{C}^*)$**. Let** $h_{t,i}$ **denote the restriction of** h_t **to** E_i**.**

- $E_{i|X \setminus D(E,\theta)}$ **are asymptotically orthogonal with respect to** h_t $(t >> 0)$**.**
- $h_{t,i|X \setminus D(E,\theta)}$ $(t >> 0)$ **are almost flat**

Theorem (rough statement)

$$\exists \lim_{t \to \infty} h_{t,i} =: h_i^{\lim} \qquad \text{on } X \setminus D(E,\theta) \text{ (after gauge transform).}$$

The limit is a flat metric.

Mazzeo-Swoboda-Weiss-Witt: $\operatorname{rank} E = 2$, $\Sigma(E,\theta)$ *smooth.*

Collier-Li: some cyclic type harmonic bundles.

$$(E, \overline{\partial}_E, t\theta, h_t) \sim \bigoplus_{i=1,2} (E_i, \overline{\partial}_{E_i}, t\phi_i, h_i^{\lim}), \quad (E, \mathbb{D}^1_{h_t}) \sim \bigoplus_{i=1,2} (E_i, \nabla_{h_i^{\lim}} + 2\,\mathrm{Re}(t\phi_i))$$

Expression of $\lim_{t\to\infty} h_{t,i}$

Preliminary **We may assume $\phi_1 = \omega$ and $\phi_2 = -\omega$, where $\omega \in H^0(X, \Omega_X^1)$, $\omega \neq 0$. We have the line bundles L_i on X and an inclusion $\iota : E \longrightarrow L_1 \oplus L_2$ such that**

- **$\iota_{|X\setminus D(E,\theta)}$ is an isomorphism, and satisfies** $\iota_{|X\setminus D(E,\theta)} \circ \theta = (\omega\,\mathrm{id}_{E_1} \oplus (-\omega)\,\mathrm{id}_{E_2}) \circ \iota_{|X\setminus D(E,\theta)}$
- **The induced morphisms $E \longrightarrow L_i$ are surjective.**

The limit $h_i^{\lim}$ should be Hermitian flat metrics on L_i which are singular at $D(E,\theta)$. Such metrics are determined by the "parabolic weights" at the points of $D(E,\theta)$.

Let L be a holomorphic line bundle on X. Let $\boldsymbol{b} = (b_P \,|\, P \in D(E,\theta)) \in \mathbb{R}^{D(E,\theta)}$ such that $\sum b_P = \deg(L)$. Then, we have a flat metric $h_L^{\boldsymbol{b}}$ of $L_{|X\setminus D(E,\theta)}$ such that

- *Let P be any point of $D(E,\theta)$. Take a local coordinate z_P such that $z_P(P) = 0$. Then, $|z_P|^{2b_P} h_L^{\boldsymbol{b}}$ gives a C^∞-metric of L around P.*

Such $h_L^{\boldsymbol{b}}$ is essentially unique.
Any flat metric of $L_{|X\setminus D(E,\theta)}$ is expressed in this way.

To describe $h_i^{\lim}$, it is enough to explain the rule to determine the parabolic weights of L_i at the points of $Z(\omega)$.

Parabolic weights **Set** $d_i := \deg(L_i)$. **We may assume** $d_1 \le d_2$.
To any $P \in D(E,\theta) = \{$**zero of** $\omega\}$, **two integers** m_P **and** ℓ_P **are attached.**

- m_P **denotes the order of zero of** ω **at** P.

  i.e., $\omega = z^{m_P} dz$ **for an appropriate coordinate** z **around** P **with** $z(P) = 0$.

- ℓ_P **denotes the length of the cokernel** $\mathrm{Cok}\big(\det(E) \longrightarrow L_1 \otimes L_2\big)$ **at** P.

Let $\chi_P : \mathbb{R}_{\ge 0} \longrightarrow \mathbb{R}_{\le 0}$ **be given as follows.**

$$\chi_P(a) := \begin{cases} (m_P+1)a - \ell_P/2 & (a \le \ell_P/2(m_P+1)) \\ 0 & (a \ge \ell_P/2(m_P+1)) \end{cases}$$

Lemma $\exists 1\ a_{E,\theta} \in \mathbb{R}$ **such that**

$$0 \le a_{E,\theta} \le \max_{P \in Z(\omega)} \big\{\ell_P/2(m_P+1)\big\}, \quad d_1 + \sum_{P \in Z(\omega)} \chi_P(a_{E,\theta}) = 0$$

Example **If** $d_1 = d_2$, **we have** $a_{E,\theta} = 0$ **and** $-\chi_P(a_{E,\theta}) = \chi_P(a_{E,\theta}) + \ell_P = \ell_P/2$.

The parabolic weights of L_1 **and** L_2 **at** P **are given by** $-\chi_P(a_{E,\theta})$ **and** $\chi_P(a_{E,\theta}) + \ell_P$.

Expression of $\lim_{t\to\infty} h_{t,i}$

We have Hermitian metrics $h^{\lim}_{L_i}$ satisfying the following conditions.

- $(L_{i|X\setminus D(E,\theta)}, h^{\lim}_{L_i})$ **are flat.**
- **For each $P \in D(E,\theta)$, take a holomorphic coordinate (U_P, z) with $z(P)=0$.**
  - $|z|^{-2\chi_P(a_{E,\theta})} h^{\lim}_{L_1}$ **is a C^∞-metric of L_1 around P.**
  - $|z|^{2\chi_P(a_{E,\theta})+2\ell_P} h^{\lim}_{L_2}$ **is a C^∞-metric of L_2 around P.**

Theorem $\lim_{t\to\infty} h_{t,i} = h^{\lim}_{L_i}$ **after gauge transform.**

How the condition for the parabolic weights appear?

We fix a Kähler metric g_X of X. Take any sequence $t_j \to \infty$.

Key step for the proof

We construct a family of Hermitian metrics $h^0_{t_j}$ of E satisfying the following condition.

- **For an appropriate $p > 1$, the L^p-norm of $\nabla_{h^0_{t_j}} \circ \nabla_{h^0_{t_j}} + |t_j|^2[\theta, \theta^\dagger_{h^0_{t_j}}]$ are bounded with respect to g_X and $h^0_{t_j}$.**

Take a neighbourhood N of $D(E,\theta)$.

- **Construction on $X \setminus N$**
- **Construction on N.**
- **Gluing**

Construction on $X \setminus N$

Take parabolic weights $\boldsymbol{b}_i := (b_{i,P} \,|\, P \in D(E,\theta)) \in \mathbb{R}^{D(E,\theta)}$ of L_i at $D(E,\theta)$.
We have the conditions

$$b_{1,P} + b_{2,P} = \ell_P, \qquad d_i - \sum_{P \in D(E,\theta)} b_{i,P} = 0 \ (i = 1,2)$$

We have the singular flat Hermitian metrics $h_{L_i}^{\boldsymbol{b}_i}$ of L_i.

Take $a > 0$. We consider a Hermitian metric

$$h_{t,\boldsymbol{b},a} := t^a h_{L_1}^{\boldsymbol{b}_1} \oplus t^{-a} h_{L_2}^{\boldsymbol{b}_2}$$

on $(L_1 \oplus L_2)_{|X \setminus D(E,\theta)} = E_{|X \setminus D(E,\theta)}$. We have

$$\nabla_{h_{1,t}} \circ \nabla_{h_{1,t}} = 0, \qquad \left[\theta, \theta^{\dagger}_{h_{1,t}}\right] = 0.$$

We shall impose some more conditions on $\boldsymbol{b}_i$ and a.

Construction on N (expression)

Around $P \in D(E,\theta)$, we take a holomorphic coordinate (U_P, z) such that $\omega = \pm d(z^{m_P+1})$.

We take local frames v_i of L_i around P satisfying the following conditions.

- $e_1 := v_1 + v_2$ **and** $e_2 := z^{\ell_P} v_2$ **give a frame of** $E_{|U_P}$**.**
- $\left| e_1 \wedge e_2 \right|_{h_{\det(E)}} = 1$**. (We fix a flat metric on** $\det(E)$**.)**

By setting $\alpha = m_P + 1$, we have a canonical expression of $t\theta$:

$$t\theta(e_1, e_2) = (e_1, e_2) \begin{pmatrix} t\alpha z^{m_P} dz & 0 \\ -2t\alpha z^{m_P - \ell_P} dz & -\alpha t z^{m_P} dz \end{pmatrix} \quad \textbf{(We have } 0 \le \ell_P \le m_P\textbf{)}$$

Construction on N (rescaling)

By rescaling, we consider the following Higgs bundles on $\mathbb{C}$, or meromorphic Higgs bundles on $(\mathbb{P}^1,\infty)$.

$$\widetilde{V}=\mathscr{O}_{\mathbb{P}^1}(*\infty)\widetilde{v}_1\oplus\mathscr{O}_{\mathbb{P}^1}(*\infty)\widetilde{v}_2,\quad \widetilde{\theta}_P(\widetilde{v}_1,\widetilde{v}_2)=(\widetilde{v}_1,\widetilde{v}_2)\begin{pmatrix}\alpha\zeta^{m_P}d\zeta & 0\\ 0 & -\alpha\zeta^{m_P}d\zeta\end{pmatrix}$$

$$\widetilde{E}_P=\langle\widetilde{e}_1,\widetilde{e}_2\rangle,\quad \widetilde{e}_1=\widetilde{v}_1+\widetilde{v}_2,\quad \widetilde{e}_2=\zeta^{\ell_P}\widetilde{v}_2$$

Because $\widetilde{\theta}_P(\widetilde{E}_P)\subset\widetilde{E}_P\otimes\Omega^1$, we obtain the Higgs bundle $(\widetilde{E}_P,\widetilde{\theta}_P)$ on $(\mathbb{P}^1,\{\infty\})$.

We take $\varphi_{P,t}:U_P\longrightarrow\mathbb{C}$ by $\varphi_{P,t}(z)=t^{1/(m_P+1)}z$.
We have $\varphi_{P,t}^*(\widetilde{E}_P,\widetilde{\theta}_P)\simeq(E,t\theta)_{|U_P}$ given by the following:

$$t^{\ell_P/2(m_P+1)}\varphi_{P,t}^*(\widetilde{e}_1)\longleftrightarrow,\ e_1,\quad t^{-\ell_P/2(m_P+1)}\varphi_{P,t}^*(\widetilde{e}_2)\longleftrightarrow,\ e_2$$

Construction on N (Kobayashi-Hitchin correspondence)

Theorem (Biquard-Boalch)

The Kobayashi-Hitchin correspondence for harmonic bundles ***with wild singularity*** **on compact Riemann surfaces.**

By using it, we would like to take a harmonic metric for $(\widetilde{E}_P, \widetilde{\theta}_P)_{|\mathbb{C}}$. We need to take parabolic weights at ∞. Let $c_{P,i}$ be the parabolic weights of $\widetilde{v}_i$.

- **Because the parabolic degree is 0, we have the condition $c_{P,1} + c_{P,2} + \ell_P = 0$.**
- **For the stability, we have $-\ell_P < c_{P,i} < 0$.**

For such $\boldsymbol{c}_P$, we have the harmonic metric $h_{t,P}^{\boldsymbol{c}_P}$ for $(\widetilde{E}_P, \widetilde{\theta}_P)_{|\mathbb{C}}$.

Recall we have $\varphi_{P,t} : U_P \longrightarrow \mathbb{C}$ and $\varphi_{P,t}^{*}(\widetilde{E}_P, \widetilde{\theta}_P) \simeq (E, t\theta)_{|U_P}$.
For the harmonic metrics $h_{t,P}^{\boldsymbol{c}_P}$, we have

$$\varphi_{P,t}^{*} h_{t,P}^{\boldsymbol{c}_P}(v_1, v_1) \sim t^{2\ell_P/2(m_P+1)} \cdot t^{2c_1/(m_P+1)}$$

$$\varphi_{P,t}^{*} h_{t,P}^{\boldsymbol{c}_P}(v_2, v_2) \sim t^{-2\ell_P/2(m_P+1)} \cdot t^{2c_2/(m_P+1)}$$

Condition for gluing

For the gluing of $h_{t,\boldsymbol{b},a}$ on $X \setminus N$ and $\varphi_{P,t}^{*} h_{t,P}^{\boldsymbol{c}}$ on N, we should have the following relations on the parameters $\boldsymbol{b} \in \mathbb{R}^{D(E,\theta)}$, $a \in \mathbb{R}$ and $\boldsymbol{c}_P \in \mathbb{R}^2$ $(P \in D(E,\theta))$,

(growth order with respect to t) $\qquad a = c_{P,1}/(m_P+1) + \ell_P/2(m_P+1)$

(comparison of the parabolic structure at P) $\qquad b_{1,P} = -c_{P,1}$

We already have the following relation:

(parabolic degree is 0) $\qquad d_1 - \sum_P b_{1,P} = 0$

Hence, we should have the following equation for a:

$$d_1 + \sum_P \Big((m_P+1)a - \ell_P/2 \Big) = 0$$

This determines $a'_{E,\theta}$.

But, $c_{P,1} = (m_P+1)a'_{E,\theta} - \ell_P/2 < 0$ are not necessarily satisfied! So, we need to modify the condition. We should replace $(m_P+1)a - \ell_P/2$ with $\chi_P(a)$.

Expectation in the higher rank case

Let $(E, \overline{\partial}_E, \theta, h)$ **be a harmonic bundle on** X **with** $\operatorname{rank} E > 2$.

(After pulling back by a ramified covering,) we have the following decomposition on $X \setminus \widetilde{D}(E,\theta)$:

$$\Sigma(E,\theta) = \bigcup_{i=1}^{m} \mathrm{Im}(\phi_i) \qquad (\phi_i \in H^0(X, \Omega^1_X))$$

Set $\widetilde{D}(E,\theta) = \bigcup_{i \neq j} \{$**zero of** $\phi_i - \phi_j\}$.

It seems natural(?) to expect

$$(E, \overline{\partial}_E, t\theta, h_t)_{|X \setminus \widetilde{D}(E,\theta)} \sim \bigoplus_{i=1}^{m} (L_i, \overline{\partial}_{L_i}, t\phi_i, h_{L_i}) \otimes (E_i, \overline{\partial}_{E_i}, \theta_i, h_{E_i})$$

- $(L_i, \overline{\partial}_{L_i}, t\phi_i, h_{L_i})$ **are harmonic bundles of rank** 1 **on** $X \setminus \widetilde{D}(E,\theta)$.
- $(E_i, \overline{\partial}_{E_i}, \theta_i, h_{E_i})$ **are complex variations of Hodge structure on** $\widetilde{D}(E,\theta)$.