

Positively ratioed representations

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- $P_\Delta = \{\text{upper triangular matrices in } G\}.$
- $\omega_{\alpha_i}(A_{i,j})_{n \times n} = \frac{1}{n} \sum_{k \leq i, l \geq i+1} A_{k,k} - A_{l,l}.$

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Proposition

Any Hitchin representation is P_Δ -positively ratioed.

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