Hilbert schemes of points and quiver varieties

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Framed sheaves on Hirzebruch surfaces

n-th Hirzebruch surface $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \subset \mathbb{P}^1 \times \mathbb{P}^2$

$$
\begin{tikzcd}
X_n = \operatorname{tot} \mathcal{O}_{\mathbb{P}^1}(-n) \arrow[r, hook, "\subset"] & \Sigma_n \arrow[d, "F"] \arrow[dr] & \\
& \mathbb{P}^1 \arrow[ul, "E"] \arrow[u, bend left, "H"] & \mathbb{P}^2
\end{tikzcd}
$$

$$H^2 = n, \qquad E^2 = -n$$

$$\operatorname{Pic}(\Sigma_n) \simeq \mathbb{Z}H \oplus \mathbb{Z}F$$

X_n is a resolution of $\mathbb{C}^2/\mathbb{Z}_n$, with $\mathbb{Z}_n$ acting as $(x, y) \rightsquigarrow (\omega x, \omega y)$

Monads for sheaves $\mathcal{E}$ in $\mathcal{M}^n(r,a,c)$
(framed on H to $\mathcal{O}_H^{\oplus r}$, normalized to $0 \le a \le r-1$)
$r = \operatorname{rk}\mathcal{E}, \quad c_1(\mathcal{E}) = aE, \quad c = c_2(\mathcal{E})$

$$0 \longrightarrow \mathcal{U} \xrightarrow{\alpha} \mathcal{V} \xrightarrow{\beta} \mathcal{W} \longrightarrow 0$$

where

$$\mathcal{U} := \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus k_1}, \ \mathcal{V} := \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus k_2} \oplus \mathcal{O}_{\Sigma_n}^{\oplus k_4}, \\ \mathcal{W} := \mathcal{O}_{\Sigma_n}(1,0)^{\oplus k_3}$$

$$k_1 = c + \frac{1}{2}na(a-1), \quad k_2 = k_1 + na, \quad k_3 = k_1 + (n-1)a, \quad k_4 = k_1 + r - a$$

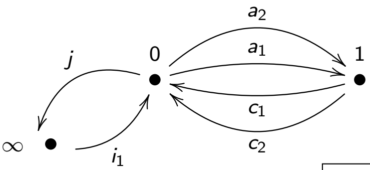
Theorem

$\mathcal{M}^n(r,a,c)$ is a smooth irreducible quasi-projective variety of dimension

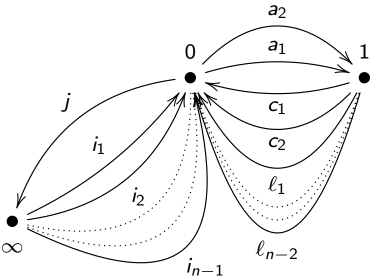
$$2r\Delta = 2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2 = 2rc + (r-1)na^2$$

It is nonempty if and only if

$$c + \tfrac{1}{2}na(a-1) \geq 0$$



$\boxed{n = 2}$



$\boxed{n \geq 3}$

For $n \geq 3$ the quiver is not a double. One could expect to have a Poisson — instead of symplectic — structure and a related moment map. However we are not able to implement this and rather explicitly fix an ideal I_n in the path algebra.

Generators:

$$\begin{cases} a_1 c_1 = a_2 c_2, \quad \text{when } n \geq 2, \\ c_1 a_1 + i_1 j = c_2 a_2, \quad a_1 c_2 = a_2 \ell_1, \quad c_2 a_1 + i_2 j = \ell_1 a_2, \quad \text{when } n \geq 3 \\ a_1 \ell_t = a_2 \ell_{t+1}, \quad \ell_t a_1 + i_{t+2} j = \ell_{t+1} a_2 \qquad \text{for} \quad t = 1, \dots, n-3, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } n \geq 4. \end{cases} \tag{1}$$

(We have not only added a framing vertex to $\mathcal{Q}_n$, but also "framed" the ideal I_n)

Define $B_n^{\mathrm{fr}} = \mathbb{C}\mathcal{Q}_n^{\mathrm{fr}} / I_n$

Definition

Fix $\vartheta \in \mathbb{R}^2$. A $(\vec{v}, w)$-dimensional representation of B_n^{fr} is said to be ϑ-semistable if, for any sub-representation $S = (S_0, S_1)$, one has:

$$\text{if } S_0 \subseteq \ker j, \text{ then } \vartheta \cdot (\dim S_0, \dim S_1) \leq 0\,;$$

$$\text{if } S_0 \supseteq \operatorname{Im} i_k \text{ for } k = 1, \dots, n-1, \qquad (*)$$

$$\text{then } \vartheta \cdot (\dim S_0, \dim S_1) \leq \vartheta \cdot (v_0, v_1)\,. \qquad (**)$$

A ϑ-semistable representation is ϑ-stable if strict inequality holds in (*) whenever $S \neq 0$ and in (**) whenever $S \neq (V_0, V_1)$.

Theorem

For every $n, c \geq 1$*, the variety* $\mathrm{Hilb}^c(X_n)$ *is isomorphic to an irreducible connected component of the quotient*

$$\operatorname{Rep}\left(B_n^{\mathrm{fr}}, \vec{v}_c, 1\right)^{ss}_{\vartheta_c} /\!/_{\vartheta_c} \mathrm{GL}_c(\mathbb{C}) \times \mathrm{GL}_c(\mathbb{C}),$$

where $\vec{v}_c = (c, c)$ *and* $\vartheta_c = (2c, -2c+1)$*.*

There are new examples of quiver varieties that are not of the Nakajima type