On Atiyah's Linear Independence Conjecture for Four Points in a Hyperbolic Plane

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Berry–Robbins Problem

Let $\mathcal{C}_n(\mathbf{R}^3)$ denote the space of configurations of n distinct ordered points in $\mathbf{R}^3$, the Euclidean 3-space.

In their study of the spin-statistic theorem in quantum mechanics, Berry and Robbins (1997) posed a very natural problem:

Berry-Robbins Problem: To construct, for each n, a continuous map

$$f_n : \mathcal{C}_n(\mathbf{R}^3) \longrightarrow U(n)/U(1)^n$$

compatible with the action of the symmetric group by permutating the points and the vectors, respectively.

A candidate solution for all n was first presented by Atiyah (2000) relying upon a certain non-degeneracy conjecture being true.

Atiyah's candidate solution

The unitary condition can be relaxed to require a map

$$F_n : \mathcal{C}_n(\mathbf{R}^3) \longrightarrow \mathrm{GL}(n, \mathbf{C})/\mathbf{C}^{*n}.$$

Given $(\mathbf{x}_1, \cdots, \mathbf{x}_n) \in \mathcal{C}_n(\mathbf{R}^3)$, this is equivalent to defining n points in $\mathbf{C}P^{n-1}$ which are linearly independent:

$$p_1, \cdots, p_n \in \mathbf{C}P^{n-1}.$$

Let us represent a point $[c_0, c_1, \cdots, c_{n-2}, c_{n-1}]$ in $\mathbf{C}P^{n-1}$ via the nonzero polynomial $c_0 t^{n-1} + c_1 t^{n-2} + \cdots, c_{n-2} t + c_{n-1}$ of degree $\leq n-1$ in a Riemann sphere variable $t \in \mathbf{C}P^1$. In homogeneous coordinates $t = [z, w]$, this is the homogeneous polynomial

$$c_0 z^{n-1} + c_1 z^{n-2} w + \cdots + c_{n-2} z w^{n-2} + c_{n-1} w^{n-1}.$$

In particular, $t - t_0$ with root $t_0 = [z_0, w_0]$ is $w_0 z - z_0 w$.

Construction of Atiyah's candidate solution

For each ordered pair $i \neq j$, we have a unit vector

$$\mathbf{v}_{ij} = \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \in S^2.$$

In particular, $\mathbf{v}_{ij} + \mathbf{v}_{ji} = \mathbf{0}$.

We identify the unit sphere S^2 with the Riemann sphere $\mathbf{C}P^1$ via a fixed stereographic projection.

Let $t_{ij} \in \mathbf{C}P^1$ be identified with $\mathbf{v}_{ij} \in S^2$ for all $i \neq j$. In particular,

$$t_{ij} \neq t_{ji}.$$

Let p_i be the polynomial in $t \in \mathbf{C}P^1$ with roots t_{ij}, $j \neq i$; that is,

$$p_i(t) = \prod_{j \neq i} (t - t_{ij}).$$

Then Atiyah's candidate map F_n is given by

$$F_n(\mathbf{x}_1, \cdots, \mathbf{x}_n) = (p_1, \cdots, p_n).$$

This map F_n will give a solution to Berry–Robbins Problem if the following conjecture of Atiyah is true.

Atiyah's Linear Independence Conjecture (2000)

For every point $(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ in the configuration space $\mathcal{C}_n(\mathbf{R}^3)$, the polynomials $p_1, \cdots, p_n$ are $\mathbf{C}$-linearly independent.

Atiyah's conjecture is clearly equivalent to that the determinant D_n of the $n \times n$ matrix with row vectors formed by the coefficients of the polynomials $p_1, \cdots, p_n$ is non-vanishing.

Invariance; Easy cases

Different choices of stereographic projections result in a $\mathrm{PSL}(2, \mathbf{C})$ change of coordinates in $\mathbf{C}P^1$.

Proposition. The linear independence in Atiyah's conjecture is preserved under a $\mathrm{PSL}(2, \mathbf{C})$ change of coordinates in $\mathbf{C}P^1$.

Atiyah noticed that his conjecture is true in the following cases:

Case **1.** $n = 2$: Trivial.
In fact, $p_1(t) = t - t_{12}$, $p_2(t) = t - t_{21}$ and $D_2 = t_{12} - t_{21} \neq 0$.

Case **2.** All the points lie on the same line: Almost trivial.
In fact, we may assume that $t_{12} = [0, 1]$ and $t_{21} = [1, 0]$. Then

$$p_i = z^{i-1} w^{n-i} = t^{i-1}, \; i = 1, \cdots, n.$$

Case **3.** $n = 3$: Non-trivial but ... easy.

Additional conjectures

To prove Atiyah's original conjecture (**Conjecture 1**), Atiyah and Sutcliffe (2002) amassed strong numerical evidence and proposed two additional conjectures.

Let $\mathbf{D}_n(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ be suitably normalized determinant.

Conjecture 2 (Atiyah–Sutcliffe) $|\mathbf{D}_n(\mathbf{x}_1, \cdots, \mathbf{x}_n)| \geq 1$.

Conjecture 3 (Atiyah–Sutcliffe)

$$|\mathbf{D}_n(\mathbf{x}_1, \cdots, \mathbf{x}_n)|^{n-2} \geq \prod_{i=1}^{n} |\mathbf{D}_{n-1}(\mathbf{x}_1, \cdots, \hat{\mathbf{x}}_i \cdots, \mathbf{x}_n)|.$$

It is east to see that

Conjecture 3 $\Longrightarrow$ **Conjecture 2** $\Longrightarrow$ **Conjecture 1**

All-but-one collinear; $n = 4$

Proposition. (Doković, 2002)
Atiyah's Conjecture 1 is true if the configuration of the n points has a reflection axis which contains at least $n - 2$ points.

This is not very difficult to prove.

The next step, $n = 4$, however, turns out to be very difficult!

Michael Eastwood and Paul Norbury (2002) gave a computer added proof for Conjecture 1 ($n = 4$) using Maple.

Mazen Bou Khuzam and Michael Johnson (2014) gave computer added proofs for Conjectures 2 and 3 ($n = 4$).

Hyperbolic version

At the very beginning, Atiyah had notice that there is a hyperbolic version for Berry–Robbins problem and a similar candidate solution.

For any two distinct points $\mathbf{x}_i$ and $\mathbf{x}_j$ in hyperbolic space $\mathbf{H}^3$, let $\mathbf{v}_{ij} \in \partial\mathbf{H}^3$ be the point that the ray running from $\mathbf{x}_i$ to $\mathbf{x}_j$ hits $\partial\mathbf{H}^3$, the boundary at infinity of $\mathbf{H}^3$.

Let $t_{ij} \in \mathbf{C}P^1$ be associated with $\mathbf{v}_{ij} \in \partial\mathbf{H}^3$ under the identification of $\partial\mathbf{H}^3$ with $\mathbf{C}P^1$ via a fixed stereographic projection. One can similarly form n polynomials of a $\mathbf{C}P^1$ variable of degree $\leq n-1$.

There are **hyperbolic versions of Conjectures 1–3** of Atiyah and Atiyah–Sutcliffe.

Hyperbolic versions of the conjectures are true in the easy cases.

Progress on hyperbolic version Conjecture 1: $n = 4$

Proposition. Atiyah's Conjecture 1 is true if some $n-1$ points of the n points in $\mathbf{H}^3$ lie on the same line.

This is not difficult to prove.

Progress on hyperbolic version Conjecture 1 for 4 points:

Joseph Malkoun (2015) obtained a human proof for Conjecture 1 (Hyperbolic 4 points: **non-planar** ...).

Jiming Ma and Ying Zhang (2016) obtain a (computer-)human proof for Conjecture 1 (Hyperbolic 4 points: **planar**).

Our Proof

Denote

$$\begin{aligned} t_{12} &= d, & t_{13} &= \infty, & t_{14} &= -D; \\ t_{21} &= -A, & t_{24} &= -C, & t_{23} &= -E; \\ t_{34} &= -B, & t_{31} &= 0, & t_{32} &= b; \\ t_{41} &= a, & t_{42} &= c, & t_{43} &= e, \end{aligned}$$

where the roman letters a, b, c, d, e and A, B, C, D, E are positive numbers. Then we have

$$\mathbf{D}_4 = \begin{vmatrix} 0 & 1 & D-d & -Dd \\ 1 & A+C+E & AC+AE+CE & ACE \\ 1 & B-b & -Bb & 0 \\ 1 & -a-c-e & ac+ae+ce & -ace \end{vmatrix}.$$

Our Proof

An easy calculation (using Maple, say) of the polynomial expansion of $\mathbf{D}_4$ gives 16 positive terms and 48 negative terms:

$$
\begin{aligned}
\mathbf{D}_4 &= Bacde + Dabce + ADace + BDacd + BDade + BDcde \\
&+ CDace + DEace + ACDbd + ACEad + ACEcd + ACEde \\
&+ ADEbd + CDEbd + ABCEd + ACDEb \\
&- abcde - Aacde - Babce - Cacde - Dabcd - Dabde \\
&- Dbcde - Eacde - ACace - ADacd - ADade - ADcde \\
&- AEace - BDabd - BDace - BDbcd - BDbde - CDacd \\
&- CDade - CDcde - CEace - DEacd - DEade - DEcde \\
&- ABDbd - ACDad - ACDcd - ACDde - ACEac - ACEae \\
&- ACEbd - ACEce - ADEad - ADEcd - ADEde - BCDbd \\
&- BDEbd - CDEad - CDEcd - CDEde - ABCDd - ABCEb \\
&- ABDEd - ACDEa - ACDEc - ACDEe - BCDEd - ABCDE.
\end{aligned}
$$

Our Proof for convex quadrilaterals

We show that in the expansion of $\mathbf{D}_4$, there are 16 negative terms whose sum with the 16 positive terms is negative, hence $\mathbf{D}_4 < 0$.

Case 1. Convex quadrilateral: $0 < \{a, b\} < c < \{d, e\}$ and $0 < \{A, B\} < C < \{D, E\}$. In this case, we have

$$\begin{aligned}
+Bacde - Cacde + Dabce - Dabde \\
+ADace - ADade + BDacd - CDacd \\
+BDade - DEcde + BDcde - CDcde \\
+CDace - CDade + DEace - DEade \\
+ACDbd - ACDcd + ACEad - ADEad \\
+ACEcd - CDEde + ACEde - ADEde \\
+ADEbd - ADEcd + CDEbd - CDEcd \\
+ABCEd - ABDEd + ACDEb - ACDEc
\end{aligned} < 0.$$

Our Proof for concave quadrilaterals

Case 2. Concave quadrilateral: $0 < a < b < c < d < e$ and $0 < A < B < C < D < E$ where we write $t_{31} = 0$, $t_{13} = \infty$,

$$a = t_{32},\ b = t_{12},\ c = t_{42},\ d = t_{41},\ e = t_{43}$$
$$-A = t_{34},\ -B = t_{14},\ -C = t_{24},\ -D = t_{21},\ -E = t_{23}.$$

In this case, we have $\mathbf{D}_4 < 0$ since

$$\begin{aligned}
&+Abcde - Dbcde + Bacde - Cbcde \\
&+ABbcd - BDbcd + ABbce - BDbce \\
&+ABbde - BDbde + BCcde - CEcde \\
&+BDcde - CDcde + BEcde - DEcde \\
&+BCDab - BCDbc + BCEab - BCEbc \\
&+BDEab - BDEbc + CDEbc - CDEce \\
&+CDEbd - CDEcd + CDEbe - CDEde \\
&+ACDEb - BCDEd + BCDEa - BCDEc \ < 0.
\end{aligned}$$

THANKS FOR YOUR ATTENTION !