Forcing with matrices of countable elementary submodels

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Joint work with Stevo Todorčević

Matrices

Definition

Let $\theta \geq \omega_2$ be a regular cardinal. By H_θ we denote the collection of all sets whose transitive closure has cardinality $< \theta$. We consider it as a model of the form $(H_\theta, \in, <_\theta)$ where $<_\theta$ is some fixed well-ordering of H_θ that will not be explicitly mentioned. The partial order $\mathcal{P}$ is the set of all functions $p : \omega_1 \to H_\theta$ satisfying:

- $\text{supp}(p) = \{\alpha < \omega_1 : p(\alpha) \neq \emptyset\}$ is a finite set;
- $p(\alpha)$ is a finite collection of isomorphic countable elementary submodels of H_θ for every $\alpha \in \text{supp}(p)$;
- for each $\alpha, \beta \in \text{supp}(p)$ if $\alpha < \beta$ then $\forall M \in p(\alpha)\ \exists N \in p(\beta)\ M \in N$;

The ordering on $\mathcal{P}$ is given by:

$$p \leq q \ \Leftrightarrow\ \forall \alpha < \omega_1 \ \ q(\alpha) \subseteq p(\alpha).$$

Matrices

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Theorem (Aspero-Mota, 2015)

$PFA^{fin}(\omega_1)$ is consistent with $\mathfrak{c} > \omega_2$.

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For $p, q \in \mathcal{P}$ we will define their 'join' $p \vee q$ as the function from ω_1 to H_θ satisfying $(p \vee q)(\alpha) = p(\alpha) \cup q(\alpha)$ for $\alpha < \omega_1$.

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We define the restriction of q to M as a function with finite support $q \mid M : \omega_1 \to H_\theta$ satisfying $\mathrm{supp}(q \mid M) = \mathrm{supp}(q) \cap \delta_M$ and for $\alpha < \delta_M$ we let $(q \mid M)(\alpha)$ to be the set of all $\varphi_{M'}(N)$ where $M' \in q(\delta_M)$, $\varphi_{M'} : M' \xrightarrow{\cong} M \cap H_\theta$ and $N \in q(\alpha) \cap M'$.

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Note that the function $q \mid M$ is in $\mathcal{P}$.

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Let $p \in \mathcal{P}$ and $\delta \in \mathrm{supp}(p)$. Then $\mathrm{cl}_\delta(p) : \omega_1 \to H_\theta$ is a function such that $\mathrm{supp}(\mathrm{cl}_\delta(p)) = \mathrm{supp}(p)$ and $\mathrm{cl}_\delta(p)(\gamma) = p(\gamma)$ for $\gamma \geq \delta$, while for $\gamma < \delta$ we define $\mathrm{cl}_\delta(p)(\gamma)$ to be the set of all $\psi_{N_1,N_2}(M)$ where $M \in p(\gamma) \cap N_1$, $N_1, N_2 \in p(\delta)$ and $\psi_{N_1,N_2} : N_1 \xrightarrow{\cong} N_2$.

Basic properties

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Let $\mathcal{G}$ be a filter generic in $\mathcal{P}$ over V, let $M, M' \prec H_{(2^\theta)^+}$ and $p, \mathcal{P} \in M \cap M'$. If $\varphi : M \xrightarrow{\cong} M'$ then for $\delta = M \cap \omega_1 = M' \cap \omega_1$ the condition $p_{MM'} = p \cup \{\langle \delta, \{M \cap H_\theta, M' \cap H_\theta\}\rangle\}$ satisfies:

$$p_{MM'} \Vdash \check{\varphi}[\dot{\mathcal{G}} \cap \check{M}] = \dot{\mathcal{G}} \cap \check{M}'.$$

Basic properties

For $p \in \mathcal{P}$, by $\bar{p}$ we define $\bar{p} : \omega_1 \to [H_{\omega_1}]^\omega$ as a function with the same support as p which maps $\alpha \in \mathrm{supp}(\bar{p})$ to the transitive collapse of some model from $p(\alpha)$, while for $\alpha \in \omega_1 \setminus \mathrm{supp}(\bar{p})$ take $\bar{p}(\alpha) = \emptyset$.

Lemma

Let $p, q \in \mathcal{P}$. If $\bar{p} = \bar{q}$, then p and q are compatible conditions.

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$\mathcal{P}$ preserves CH.

Kurepa tree I

Definition

A tree $\langle T, < \rangle$ is called Kurepa tree if is of height ω_1, with at least ω_2 branches and with all levels countable.

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- Introduced in Kurepa's PhD thesis (Paris, 1935);
- Solovay showed that there is a Kurepa tree in L;
- Silver showed that consistently there are no Kurepa trees;
- Devlin showed that all the theories ZFC$\pm$CH$\pm$SH$\pm$KH are consistent.

Kurepa tree II

Theorem

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Sketch of the proof:
Consider the family of functions $f_\alpha : \omega_1 \to \omega_1$ $(\alpha < \omega_2)$ defined by

$$f_\alpha(\delta) = \begin{cases} \xi, & \text{if there is } M \in G(\delta) \text{ such that } \alpha \in M \text{ and } \pi_M(\alpha) = \xi, \\ 0, & \text{otherwise.} \end{cases}$$

Denote this family by $\mathcal{F} = \{f_\alpha : \alpha < \omega_2\}$ and for a fixed $\alpha < \omega_2$ let the α-th branch of our tree be given by $\mathcal{F}_\alpha = \{f_\alpha \restriction \delta : \delta < \omega_1\}$.

So the Kurepa tree will be $T = \bigcup_{\delta < \omega_1} T_\delta$, where $T_\delta = \{f_\alpha \restriction \delta : \alpha < \omega_2\}$ are its levels. □

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Corollary

Every uncountable downward closed set $S \subseteq T$ contains a branch of T.

Continuous matrices

Definition

Let $\mathcal{P}_c$ be the suborder of $\mathcal{P}$ containing all the conditions satisfying:

- for every $p \in \mathcal{P}_c$ there is a continuous $\in$-chain $\langle M_\xi : \xi < \omega_1 \rangle$ (i.e. if β is a limit ordinal, then $M_\beta = \bigcup_{\xi<\beta} M_\xi$) of countable elementary submodels of H_θ such that $\forall \xi \in \operatorname{supp}(p) \quad M_\xi \in p(\xi)$.

Almost Souslin Kurepa tree I

Definition

A tree S of height ω_1 is an almost Souslin tree if for every antichain $X \subseteq S$, the set (S_γ is the γ-th level of S)

$$L(X) = \{\gamma < \omega_1 : X \cap S_\gamma \neq \emptyset\}$$

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- The existence of an almost Souslin Kurepa tree was asked by Zakrzewski (1986).
- Todorcevic showed that the existence of an almost Souslin Kurepa tree is consistent (1987).
- Golshani showed there is an almost Souslin Kurepa tree in L (2013).

Almost Souslin Kurepa tree II

Theorem (CH)

The tree T is an almost Souslin Kurepa tree.

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Sketch of the proof: Let $\tau \in H_\theta$ be a $\mathcal{P}_c$-name. Then the set

$$\Gamma_\tau = \{\gamma < \omega_1 : \exists M \in G_c(\gamma)\ \tau \in M\ \&\ M[\mathcal{G}_c] \cap \omega_1 = M \cap \omega_1 = \gamma\}.$$

is closed and unbounded in ω_1.

Now, let τ' be a $\mathcal{P}_c$-name for an antichain X in T.

Because CH holds in V, $\mathcal{P}_c$ is ω_2-c.c. so there is a $\mathcal{P}_c$-name σ for X which is in H_θ.

Then $L(X) \cap \Gamma_\sigma = \emptyset$. $\square$