Boolean valued second order logic

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In this talk...

We will work on 2nd-order logic.

From now on...

We work in ZFC.

Review; 1st-order logic

1st-order logic enjoys several nice properties:

- Completeness Theorem
- The set of valid sentences is Σ^0_1.
- Compactness Theorem
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How about 2nd-order logic?

2nd-order logic; Two semantics

1. Henkin semantics: Very simple (very week), enjoys completeness, compactness.
2. Full semantics: Highly complex (very powerful), does not enjoy completeness, compactness.

Henkin semantics

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Example

A 2nd-order structure $M = (A, \mathcal{P}(A), \ldots)$ is called a *full 2nd-order structure*.

Henkin semantics ctd.

Theorem (Henkin)

Henkin semantics is sound and complete to a standard syntax in 2nd-order logic.

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Henkin semantics is essentially the same as the standard semantics for 1st-order logic.

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Note: One cannot expect a completeness result for full semantics with 'simple' syntax.

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Note: When $\mathbb{B} = \{0, 1\}$, it is the same as considering all the subsets $\mathcal{P}(A)$, i.e., full semantics.

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1. A is a nonempty set,
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3. for each n-ary relational symbol R_i in $\mathcal{L}$, $R_i^M \colon A^n \to \mathbb{B}$.

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Example

If $\mathbb{B} = \{0, 1\}$, R_i^M is a relation in 1st-order logic and M is the same as a 1st-order structure.

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Let $M = (A, \mathbb{B}, \{R_i\})$ be a Boolean-valued $\mathcal{L}$-structure. Then we assign $\|\phi[\vec{a}, \vec{f}]\|^M$ to each 2nd-order formula ϕ, $\vec{a} \in A^{<\omega}$, and $\vec{f} \in (A \to \mathbb{B})^{<\omega}$ as follows:

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1. ϕ is $R_i(\vec{y})$. Then $\|R_i(\vec{y})[\vec{a}]\|^M = R_i^M(\vec{a})$.
2. ϕ is $X(y)$. Then $\|X(y)[a, f]\|^M = f(a)$.
3. Boolean combinations are as usual.
4. ϕ is $\exists y \psi$. Then $\|\exists y \psi[\vec{a}, \vec{f}]\|^M = \bigvee_{b \in A} \|\psi[b, \vec{a}, \vec{f}]\|^M$.
5. ϕ is $\exists X \psi$. Then $\|\exists X \psi[\vec{a}, \vec{f}]\|^M = \bigvee_{g\colon A \to \mathbb{B}} \|\psi[\vec{a}, g, \vec{f}]\|^M$.

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$$0^{2b} = \{\phi \mid \phi \text{ is Boolean valid}\}.$$

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Answer

One canNOT decide in ZFC!

Boolean-valued semantics vs full semantics

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If you assume Large Cardinals and the Ω-Conjecture, then 0^{2b} is strictly simpler than 0^{2f}.

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Let ϕ be a Π_2-sentence in set theory.
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Main interest: $0^\Omega = \{\phi \mid \phi \text{ is } \Omega\text{-valid}\}$.

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4. (Woodin) If you assume Large Cardinals, then every sentence in the 2nd-order arithmetic true in V is in 0^Ω.

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Phenomenon: The stronger axioms of infinity you assume, the more sentences belong to 0^Ω.

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The Ω-Conjecture

Under the existence of large cardinals,

$$\text{“}\Omega\text{-provability”} = \text{“}\Omega\text{-validity”}.$$

Boolean-valued 2nd-order logic and Ω-logic

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Key Lemma

$$0^{2b} \equiv_{\mathrm{T}} 0^{\Omega}.$$

Point: "Considering all the Boolean-valued subsets of A" $=$ "considering all the subsets of A in any set generic extension".

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Theorem (Woodin)

Assuming large cardinals and the Ω-Conjecture, 0^{Ω} is Δ_2 in the language of set theory.

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Corollary

Assuming large cardinals and the Ω-Conjecture, then 0^{2b} is strictly simpler than 0^{2f}.

Löwenheim Skolem number

Definition

Let $\mathcal{L}$ be a logic. Then the Löwenheim-Skolem number of $\mathcal{L}$ ($\ell(\mathcal{L})$) is defined as follows:

$$\ell(\mathcal{L}) = \text{min}\{\kappa \mid \text{if an } \mathcal{L}\text{-sentence } \phi \text{ has a model, then it has a model of size at most } \kappa\}$$

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Example

1. If FOL is 1st-order logic with the standard semantics, then $\ell(\text{FOL}) = \omega$.
2. If SOL is 2nd-order logic with full semantics, then

$$\ell(\text{SOL}) = \text{sup}\{\alpha \mid \alpha \text{ is } \Delta_2\text{-definable in the language of set theory}\}.$$

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One can obtain similar results for Hanf number for 2nd-order logic, and the compactness number for infinitary 2nd-order logic.

Conclusion

- Henkin semantics and full semantics are the major semantics for 2nd-order logic.
- Henkin semantics is essentially the same as 1st-order logic.
- full semantics is much more powerful than 1st-order logic and highly complicated.
- Boolean-valued semantics is a powerful semantics for 2nd-order logic while it could be simpler and easier to deal with than full semantics depending on set-theoretic assumptions.

The End.