The proof-theoretic strength of RT^2_2

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New Challenges in Reverse Mathematics
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Main question

Question (Cholak/Jockusch/Slaman 2001)

What is the proof-theoretic strength, or provably total functions (in other words, Π^0_2-part) of RT^2_2?

There are so many studies of the strength of RT^2_2.

Theorem (Hirst 1987)

$\mathsf{RCA}_0 + \mathrm{RT}^2_2$ *implies* $\mathrm{B}\Sigma^0_2$.

Theorem (Cholak/Jockusch/Slaman 2001)

$\mathsf{WKL}_0 + \mathrm{RT}^2_2 + \mathrm{I}\Sigma^0_2$ *is a* Π^1_1*-conservative extension of* $\mathsf{RCA}_0 + \mathrm{I}\Sigma^0_2$.

Thus, the first order strength of RT^2_2 is in between $\mathrm{B}\Sigma^0_2$ and $\mathrm{I}\Sigma^0_2$. Note that $\mathrm{B}\Sigma^0_2$ is a Π^0_2-conservative extension of PRA, while $\mathrm{I}\Sigma^0_2$ is strictly stronger.

Main question

Recently, there are several important improvements.

Theorem (Chong/Slaman/Yang 2014)

$\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *does not imply* $\mathrm{I}\Sigma^0_2$.

Theorem (Chong/Kreuzer/Yang 2015)

$\mathsf{WKL}_0 + \mathrm{SRT}^2_2$ *is* Π^0_3*-conservative over* $\mathsf{RCA}_0 + \mathrm{WF}(\omega^\omega)$.

Here is our main result.

Theorem (Patey/Y)

$\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *is a* $\tilde{\Pi}^0_3$*-conservative extension of* RCA_0.
(Here $\tilde{\Pi}^0_n$*-formula is of the form* $\forall X \theta$ *where* θ *is* Π^0_n*.)*

This is a optimal conservation result over RCA_0 since there is a Σ^0_3-consequence of $\mathsf{RCA}_0 + \mathrm{RT}^2_2$ which is not provable in RCA_0.

Outline

1. Density, α-largeness and $\tilde{\Pi}^0_3$-conservation
   - PH^2_2 and density
   - Conservation via density
   - Decomposition of density by α-largeness
2. Bounding ω^k-large(RT^2_2) sets
   - Decomposition by $\mathrm{RT}^2_2 = \mathrm{ADS} + \mathrm{EM}$
   - Bounding α-large(ADS) sets
   - Bounding α-large(EM) sets by the finite grouping principle
3. The strength of the grouping principle
   - Infinite grouping principle
   - Computability theoretic strength of GP^2_2
   - Conservation for GP^2_2

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Ramsey's theorem and its finite approximation

An important finite consequence of Ramsey's theorem is the following Paris-Harrington principle.

Theorem (PH^2_2)

For any $X_0 \subseteq_{\mathrm{inf}} \mathbb{N}$, there exists $F \subseteq_{\mathrm{fin}} X_0$ such that for any $f : [F]^2 \to 2$ there exists $H \subseteq F$ such that H is homogeneous for f and H is relatively large, i.e., $|H| > \min H$.

- PH^2_2 is an easy consequence of $\mathrm{WKL}_0 + \mathrm{RT}^2_2$.
- Actually, we can prove it just within RCA_0.
- The $\tilde{\Pi}^0_3$-part of (infinite) Ramsey's theorem is characterized by "iterated version" Paris-Harrington-like principles.

Ramsey's theorem and its finite approximation

Definition (RCA_0)

- A finite set $X \subseteq \mathbb{N}$ is said to be 0-*dense* if $|X| > \min X$.
- A finite set X is said to be $m+1$-*dense* if for any $P : [X]^2 \to 2$, there exists $Y \subseteq X$ which is m-dense and P-homogeneous.

Note that "X is m-dense(n, k)" can be expressed by a Σ^0_0-formula.

Definition

- $m\mathrm{PH}^2_2$: for any $X_0 \subseteq_{\mathrm{inf}} \mathbb{N}$, there exists $F \subseteq_{\mathrm{fin}} X_0$ such that F is m-dense.

Note that $m\mathrm{PH}^2_2$ is still a consequence of $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ for any $m \in \omega$.

Conservation via density

By a simple generalization of indicator arguments we have the following.

Theorem (A generalization of Bovykin/Weiermann)

$\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *is a* $\tilde{\Pi}^0_3$*-conservative extension of* $\mathsf{RCA}_0 + \{m\mathrm{PH}^2_2 \mid m \in \omega\}$.

Thus, to prove the main theorem, what we need is the following.

WANT

For each $m \in \omega$, prove $m\mathrm{PH}^2_2$ within RCA_0.

Since *m*-dense sets are very complicated, we will decompose the density notion.

α-large sets

We want to bound the size of m-dense sets.
For that, we use a tool from proof theory.

Definition

For ordinals below ω^ω (with a fixed primitive recursive ordinal notation),

- X is said to be $\alpha + 1$-large if $X - \{\min X\}$ is α-large,
- X is said to be γ-large if $X - \{\min X\}$ is $\gamma[\min X]$-large (γ: limit), where $\alpha + \omega^k[x] = \alpha + \omega^{k-1} \cdot x$.

- X is m-large if $|X| \geq m$.
- X is ω-large if $|X| \geq \min X$, *i.e.*, relatively large.
- X is ω^{k+1}-large if $X - \{\min X\}$ splits up into $\min X$ many ω^k-large sets.

PH^2_2 with α-large sets

Definition

X is said to be *α-large*(RT^2_k) if for any $P : [X]^2 \to k$, there exists $Y \subseteq X$ which is α-large and P-homogeneous.

Here is an important result connecting α-largeness and PH.

Theorem (Solovay/Katonen 1981)

X is $\omega^{k+3} + \omega^3 + k + 4$-large $\Rightarrow$ X is ω-large(RT^2_k).

Thus, any ω^6-large set (with min > 3) is ω-large(RT^2_2), which is 1-dense. (In what follows, we only consider finite sets with thier min > 3.)

Proposition

For any $k \in \omega$, $\mathrm{RCA}_0 \vdash$ "any infinite set contains ω^k-large set."

Thus, $1\mathrm{PH}^2_2$ is provable in RCA_0.

We want to generalize the previous situation.

WANT

For any $k \in \omega$, find $n \in \omega$ so that RCA_0 proves

- X is ω^n-large $\Rightarrow$ X is ω^k-large(RT^2_2).

This is enough to prove $m\mathrm{PH}^2_2$ within RCA_0 by the following argument.

- ω^6-large $\Rightarrow$ ω-large(RT^2_2) $\Rightarrow$ 1-dense.
- Take $n_2 \in \omega$ so that ω^{n_2}-large $\Rightarrow$ ω^6-large(RT^2_2).
  Then, ω^{n_2}-large $\Rightarrow$ 2-dense.
- Take $n_3 \in \omega$ so that ω^{n_3}-large $\Rightarrow$ ω^{n_2}-large(RT^2_2).
  Then, ω^{n_3}-large $\Rightarrow$ 3-dense.
- $\vdots$

Thus, for any $m \in \omega$, there exists $n \in \omega$ such that

$$\omega^n\text{-large} \Rightarrow m\text{-dense}.$$

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Decomposition by $\mathrm{RT}^2_2 = \mathrm{ADS} + \mathrm{EM}$

RT^2_2 can be decomposed into $\mathrm{ADS} + \mathrm{EM}$ by using the idea of transitive coloring (Shore/Hirschfeldt and Bovykin/Weiermann).

Definition

- X is *α-large*(ADS) if for any transitive $P : [X]^2 \to 2$, there exists $Y \subseteq X$ which is α-large and P-homogeneous.
- X is *α-large*(EM) if for any $P : [X]^2 \to 2$, there exists $Y \subseteq X$ which is α-large such that P is transitive on $[Y]^2$.

Now, what we need are

WANT

For any $k \in \omega$, find $n_1, n_2 \in \omega$ so that RCA_0 proves

- X is ω^{n_1}-large $\Rightarrow$ X is ω^k-large(ADS).
- X is ω^{n_2}-large $\Rightarrow$ X is ω^k-large(EM).

Bounding α-large(ADS) sets

Thanks to the transitivity, we can calculate the size of the above sets directly.

Lemma

X is ω-large$(\mathrm{RT}^2_{2k+2}) \Rightarrow X$ is ω^k-large(ADS).

Then, by Solovay/Ketonen's theorem, we have

Theorem

X is ω^{2k+6}-large $\Rightarrow X$ is ω^k-large(ADS).

Thus,

Theorem

For any $k \in \omega$, there exists $n \in \omega$ such that RCA_0 proves

- *X is ω^n-large $\Rightarrow X$ is ω^k-large(ADS).*

Bounding α-large(EM) sets

WANT

For any $k \in \omega$, find $n \in \omega$ so that RCA_0 proves

- X is ω^n-large $\Rightarrow$ X is ω^k-large(EM).

Constructing a "large" solution for EM is rather difficult.
By Solovay/Ketonen's theorem, we can always construct ω-large solutions. Then, how can we construct ω^2-large solution from them?

$\Rightarrow$ want to combine "ω-large many" ω-large solution.

- If $f : [\mathbb{N}]^2 \to 2$ is transitive on $[F_1]^2$ and $[F_2]^2$, then what is needed to say that f is transitive on $[F_1 \cup F_2]^2$?

  $\Rightarrow$ "$\exists c \in 2\, \forall x \in F_1\, \forall y \in F_2\, f(x, y) = c$" is enough.

The following "grouping principle" is essential to use this idea.

The grouping principle (finite version)

Definition (RCA_0)

Let $n, k \in \omega$.
Given $f : [X]^2 \to 2$, an (ω^n, ω^k)*-grouping for* f is a finite family of finite sets $\langle F_i \subseteq X \mid i < l \rangle$ such that

- $\forall i < j < l \max F_i < \min F_j$,
- $\forall i < l\, F_i$ is ω^n-large,
- $\{\max F_i \mid i < l\}$ is ω^k-large, and,
- $\forall i < j\, \exists c < 2\, \forall x \in F_i, \forall y \in F_j\, f(x, y) = c$.

$\mathrm{FGP}^2_2(\omega^n, \omega^k)$: for any $X_0 \subseteq_{\mathrm{inf}} \mathbb{N}$, there exists $X \subseteq_{\mathrm{fin}} X_0$ such that for any $f : [X]^2 \to 2$, there exists an (ω^n, ω^k)-grouping for f.

Theorem (we will see this in the next section.)

For any $n, k \in \omega$, $\mathsf{RCA}_0 \vdash \mathrm{FGP}^2_2(\omega^n, \omega^k)$.

Bounding FGP^2_2

We can give an "ω^m-large type bound" for the finite grouping by the following theorem.

Theorem (Generalized Parsons theorem)

Let $\psi(F)$ be a Σ^0_1-formula with exactly the displayed free variables. Assume that

$$\mathrm{RCA}_0 \vdash \forall X \subseteq \mathbb{N}(X \text{ is infinite} \to \exists F \subseteq_{\mathrm{fin}} X\psi(F)).$$

Then, there exists $n \in \omega$ such that

$$\mathrm{RCA}_0 \vdash \forall Z \subseteq_{\mathrm{fin}} \mathbb{N}(Z \text{ is } \omega^n\text{-large} \to \exists F \subseteq Z\psi(F)).$$

Corollary

For any $n, k \in \omega$, there exists $m \in \omega$ such that RCA_0 proves

- *X is ω^m-large $\Rightarrow$ any coloring $f : [X]^2 \to 2$ has an (ω^n, ω^k)-grouping for f.*

Bounding α-large(EM) sets by FGP^2_2

Now we bound α-large(EM) sets inductively.
Assume (within RCA_0) X is ω^{n_k}-large $\Rightarrow$ X is ω^k-large(EM).
(The case $k = 1$, $n_1 = 6$ is good by Solovay/Ketonen's theorem.)
Then, $\mathrm{FGP}^2_2(\omega^{n_k}, \omega^6)$ gives a bound for ω^k-large(EM) sets.

- Take $n_{k+1} \in \omega$ so that any $\omega^{n_{k+1}}$-largeness bounds $\mathrm{FGP}^2_2(\omega^{n_k}, \omega^6)$.
- Given $f : [X]^2 \to 2$, there exists an (ω^{n_k}, ω^6)-grouping $\langle F_i \mid i < l \rangle$ for f.
- Since each of F_i is ω^{n_k}-large, there exists ω^k-large sets $H_i \subseteq F_i$ such that f is transitive on $[H_i]^2$.
- Since $\{\max F_i \mid i < l\}$ is ω^6-large, there exists $\bar{H} \subseteq \{0, \dots, l-1\}$ such that $\{\max F_i \mid i \in \bar{H}\}$ is ω-large and f-homogeneous.
- Thus, $\exists c < 2$, $\forall i, j \in \bar{H}$, $i \neq j$, $\forall x \in F_i \, \forall y \in F_j$, $f(x, y) = c$.

Bounding α-large(EM) sets by FGP^2_2

- $H = \bigcup_{i \in \bar{H}} H_i$ is ω^{k+1}-large, since each of H_i is ω^k-large and $|\bar{H}| > \min\{\max F_i \mid i \in \bar{H}\} > \min H$.
- f is transitive on $[H]^2$, since for $x, y, z \in H$,
  - if $x, y, z \in H_i$ then ok since f is transitive on $[H_i]^2$,
  - if $x \in H_i$ and $y, z \in H_j$ then $f(x, y) = f(x, z)$, similar for the case $x, y \in H_i$ and $z \in H_j$,
  - if x, y, z are in different groups, then $f(x, y) = f(y, z) = f(x, z) = c$.
- Thus, we have "$\omega^{n_{k+1}}$-large $\Rightarrow$ X is ω^{k+1}-large(EM)"

Theorem

For any $k \in \omega$, there exists $n \in \omega$ such that RCA_0 proves

- *X is ω^n-large $\Rightarrow$ X is ω^k-large(EM).*

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To prove the finite grouping principle...

WANT

Prove $\mathrm{FGP}^2_2(\omega^n, \omega^k)$ for any $n, k \in \omega$ within RCA_0.

- FGP^2_2 is a too complicated finite combinatorics and thus analyzing this within RCA_0 directly is hard.
- Instead of proving FGP^2_2 directly, we will consider infinite combinatorial principle which implies FGP^2_2.

  $\Rightarrow$ Go back to infinite combinatorics.

Infinite grouping principle

Definition (RCA_0)

Let $\alpha < \omega^\omega$.
Given $f : [\mathbb{N}]^2 \to 2$, an infinite *α-grouping for f* is a infinite family of finite sets $\langle F_i \subseteq X \mid i \in \mathbb{N}\rangle$ such that

- $\forall i < j, \max F_i < \min F_j$,
- $\forall i \in \mathbb{N}\; F_i$ is α-large,
- $\forall i < j\, \exists c < 2\, \forall x \in F_i, \forall y \in F_j\; f(x, y) = c$.

$\mathrm{GP}^2_2(\alpha)$: for any $f : [\mathbb{N}]^2 \to 2$, there exists an infinite α-grouping for f.

Note that we can generalize GP to versions for n-tuples, k-pairs and for wider/abstract largeness notions.

Computability theoretic strength of GP^2_2

$\mathrm{GP}^2_2(2)$ is already non-trivial, moreover, we can see the following.

Theorem (RCA_0)

$\mathrm{GP}^2_2(\omega)$ *implies rainbow Ramsey theorem for pairs.*

As the usual analysis for Ramsey-type statement, considering the grouping principle for stable colorings (SGP^2_2) is useful.

Proposition (RCA_0)

$\mathrm{COH} + \mathrm{SGP}^2_2 \rightarrow \mathrm{GP}^2_2$.

Then, one can construct a solution of SGP^2_2 by a version of Mathias forcing.

Computability theoretic strength of GP^2_2

Theorem

- $\mathrm{SGP}^2_2(\omega^n)$ *has an* ω*-model with only low sets.*
- $\mathrm{SGP}^2_2(\omega^n)$ *can preserve countably many hyperimmune sets.*

Corollary

- $\mathsf{WKL}_0 + \mathrm{SGP}^2_2 + \mathrm{SADS}$ *does not imply* SRT^2_2, SEM *or* COH.
- $\mathsf{WKL}_0 + \mathrm{GP}^2_2 + \mathrm{EM}$ *does not imply* ADS.

- One can often transform a low solution construction into a construction of a solution preserving $\mathrm{I}\Sigma^0_1$ in nonstandard models.

  $\Rightarrow$ Can we use this for a conservation proof?

WANT

Prove $\mathrm{FGP}^2_2(\omega^n, \omega^k)$ for any $n, k \in \omega$ within RCA_0.

We will show this by proving $\tilde{\Pi}^0_3$-conservation for $\mathrm{WKL}_0 + \mathrm{GP}^2_2(\omega^n)$ over RCA_0. By transforming the previous low solution construction,

Theorem

Let $(M, S) \models \mathrm{B}\Sigma^0_2$ and $f : [M]^2 \to 2$ is a stable coloring, then there exists $G \subseteq M$ such that

$$(M, S \cup \{G\}) \models \mathrm{I}\Sigma^0_1 + \text{“}G \text{ is an infinite } \omega^n\text{-grouping for } f.\text{”}$$

Conservation for $\mathrm{B}\Sigma^0_2$ vs $\mathrm{I}\Sigma^0_1$

The previous solution construction cannot be repeated. However, we can still derive $\tilde{\Pi}^0_3$-conservation.

Theorem

Let Γ be a formula of the form $\forall X \exists Y \theta(X, Y)$ where θ is Π^0_2. Then, $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2 + \Gamma$ is a $\tilde{\Pi}^0_3$-conservative extension of $\mathrm{I}\Sigma^0_1$ if the following condition holds:

($\dagger$) *for any countable recursively saturated model $(M, S) \models \mathrm{B}\Sigma^0_2$ and for any $X \in S$, there exists $Y \subseteq M$ such that $(M, S \cup \{Y\}) \models \mathrm{I}\Sigma^0_1 + \theta(X, Y)$.*

We have seen that $\mathrm{SGP}^2_2(\omega^n)$ satisfies this.
Note that WKL and ADS, which implies COH, also satisfy this condition.

Conservation for GP^2_2 and FGP^2_2

Corollary

$\mathsf{WKL}_0 + \mathrm{GP}^2_2(\omega^n)$ *is a* $\tilde{\Pi}^0_3$*-conservative extension of* RCA_0.

By the compactness argument, we can easily see the following.

Theorem

For any $n, k \in \omega$, $\mathsf{WKL}_0 + \mathrm{GP}^2_2(\omega^n) \vdash \mathrm{FGP}^2_2(\omega^n, \omega^k)$.

Thus, we have

Corollary

For any $n, k \in \omega$, $\mathrm{FGP}^2_2(\omega^n, \omega^k)$ *is provable within* RCA_0.

Theorem (Patey/Y)

$\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *is a* $\tilde{\Pi}^0_3$*-conservative extension of* RCA_0*,*
thus, it is a Π^0_2*-conservative extension of* PRA.

Corollary

$\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *does not imply the consistency of* $\mathrm{I}\Sigma^0_1$ *nor the totality of Ackermann function.*

The proof of the theorem can be formalizable within WKL_0. Thus, we have the following.

Corollary

PRA *proves* $\mathrm{Con}(\mathsf{PRA}) \leftrightarrow \mathrm{Con}(\mathsf{WKL}_0 + \mathrm{RT}^2_2)$.

Questions

Two big questions.

Question

- Is $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ Π^1_1-conservative over $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$?
- Is there a significant speed-up between RCA_0 and $\mathsf{WKL}_0 + \mathrm{RT}^2_2$?

Many smaller questions.

Question

- Does $\mathrm{GP}^2_2(\omega^n)$ imply $\mathrm{B}\Sigma^0_2$ or EM?
- Does ADS or EM imply $\mathrm{GP}^2_2(\omega^n)$?
- Do $\mathrm{GP}^2_2(\omega^n)$'s form a strict hierarchy?
- What is the strength of GP^n_k in general?

Thank you!

- Ludovic Patey and Y, The proof-theoretic strength of Ramsey's theorem for pairs and two colors, draft, available at `http://arxiv.org/abs/1601.00050`