Thoughts on indicators and density notions

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Aim

Introducing...

- basic ideas of indicators and their (slight) generalization,
- several consequences of the indicator arguments,
- some conservation results of combinatorial principles.

"Indicators are useful to analyze the first-order part of combinatorial statements in second-order arithmetic."

Nonstandard models of arithmetic

In this talk we will mainly use the base system EFA $= \mathrm{I}\Delta_0 + \exp$ or RCA_0^*, which consists of $\mathrm{I}\Delta_0^0 + \exp$ plus Δ_1^0-comprehension, and models we will consider will be countable nonstandard.
Let $M \models$ EFA.

- $I \subseteq M$ is said to be a cut (abbr. $I \subseteq_e M$) if $a < b \in I \to a \in I$ and I is closed under addition $+$ and multiplication $\cdot$.
- $\mathrm{Cod}(M) = \{X \subseteq M \mid X \text{ is } M\text{-finite}\}$, where M-finite set is a set coded by an element in M (by means of the usual binary coding).
- for $Z \in \mathrm{Cod}(M)$, $|Z|$ denotes the internal cardinality of Z in M.
- for $I \subseteq_e M$, $\mathrm{Cod}(M/I) := \{X \cap I \mid X \in \mathrm{Cod}(M)\}$.

Proposition

If $I \subseteq_e M$, then I is a Σ_0-elementary substructure of M.

Cuts

There are several important types of cuts.

Theorem (exponentially closed cut, Simpson/Smith)

Let $M \models \mathrm{EFA}$, *and let* $I \subsetneq_e M$. *Then the following are equivalent.*

1. $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0^*$.
2. *I is closed under* exp.

Theorem (semi-regular cut)

Let $M \models \mathrm{EFA}$, *and let* $I \subsetneq_e M$. *Then the following are equivalent.*

1. $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0$.
2. *I is semi-regular, i.e., if* $X \in \mathrm{Cod}(M)$ *and* $|X| \in I$, *then* $X \cap I$ *is bounded in* I.

Cuts

Theorem (strong cut)

Let $M \models \mathrm{EFA}$, and let $I \subsetneq_e M$. Then the following are equivalent.

1. $(I, \mathrm{Cod}(M/I)) \models \mathrm{ACA}_0$.
2. *I is strong, i.e., if for any $a > I$ for any $b \in I$ and for any $f : [[0, a]]^3 \to b$ coded in M, there exists $Y \subseteq [0, a]$ such that Y is f-homogeneous and $Y \cap I$ is unbounded in I.*

These combinatorial characterization of cuts play key roles in the definition of indicators.

Indicators

Let T be a theory of second-order arithmetic.
A Σ_0-definable function $Y : [M]^2 \to M$ is said to be an *indicator* for $T \supseteq \mathrm{WKL}_0^*$ if

- $Y(x, y) \leq y$,
- if $x' \leq x < y \leq y'$, then $Y(x, y) \leq Y(x', y')$,
- $Y(x, y) > \omega$ if and only if there exists a cut $I \subseteq_e M$ such that $x \in I < y$ and $(I, \mathrm{Cod}(M/I)) \models T$.

  (Here, $Y(x, y) > \omega$ means that $Y(x, y) > n$ for any standard natural number n.)

Example

- $Y(x, y) = \max\{n : \exp^n(x) \leq y\}$ is an indicator for WKL_0^*.
- $$Y(x, y) = \max\{n : \text{any } f[[x, y]]^n \to 2 \text{ has a homogeneous set } Z \subseteq [x, y] \text{ such that } |Z| > \min Z\}$$

  is an indicator for ACA_0.

Basic properties of indicators

Theorem

If Y is an indicator for a theory T, then for any $n \in \omega$,

$$T \vdash \forall x \exists y Y(x, y) \geq n.$$

Theorem

If Y is an indicator for a theory T, then, T is a Π^0_2-conservative extension of $\mathrm{EFA} + \{\forall x \exists y Y(x, y) \geq n \mid n \in \omega\}$*.*

Let $F^Y_n(x) = \min\{y \mid Y(x, y) \geq n\}$.

Theorem

If Y is an indicator for a theory T and $T \vdash \forall x \exists y \theta(x, y)$ for some Σ_1-formula θ, then, there exists $n \in \omega$ such that

$$T \vdash \forall x \exists y < F^Y_n(x) \theta(x, y).$$

Set indicators

Let T be a theory of second-order arithmetic.
A Σ_0-definable function $Y : \mathrm{Cod}(M) \to M$ is said to be a *set indicator* for $T \supseteq \mathsf{WKL}_0^*$ if

- $Y(F) \leq \max F$,
- if $F \subseteq F'$, then $Y(F) \leq Y(F')$,
- $Y(F) > \omega$ if and only if there exists a cut $I \subseteq_e M$ such that $\min F \in I < \max F$ and $(I, \mathrm{Cod}(M/I)) \models T$, and $F \cap I$ is unbounded in I.

Note that if Y is a set indicator, then $Y'(x, y) = Y([x, y])$ is an indicator function.

Example

- $Y(F) = \max\{m : F \text{ is } m\text{-dense}(\mathrm{RT}^2_2)\}$ is an indicator for $\mathsf{WKL}_0 + \mathrm{RT}^2_2$.

Actually, density notions provide set indicators for many theories.

Ramsey-like statements

Definition (Ramsey-like formulas)

A *Ramsey-like-Π^1_2-formula* is a Π^1_2-formula of the form

$$(\forall f : [\mathbb{N}]^n \to k)(\exists Y)(Y \text{ is infinite} \land \Psi(f, Y))$$

where $\Psi(f, Y)$ is of the form $(\forall G \subseteq_{\mathrm{fin}} Y)\Psi_0(f \restriction [[0, \max G]_{\mathbb{N}}]^n, G)$ such that Ψ_0 is a Δ^0_0-formula.

(Here, $n, k \in \omega$ or they are unbounded parameters.)

- In particular, RT^n_k is a Ramsey-like-Π^1_2-statement where $\Psi(f, Y)$ is the formula "Y is homogeneous for f".
- Any Π^1_2-formula of the form $\forall X \exists Y \Theta(X, Y)$ where Θ is a Σ^0_3-formula is equivalent to a Ramsey-like formula over WKL_0.

A Ramsey-like statement has an indicator given by the density notion.

Density

Definition (EFA, Density notion)

Given a Ramsey-like formula

$$\Gamma = (\forall f : [\mathbb{N}]^n \to k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y)),$$

- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be 0-*dense*(Γ) if $|Z|, \min Z > 2$,
- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be $(m+1)$-*dense*(Γ) if
  - (for any $n, k < \min Z$ and) for any $f : [[0, \max Z]]^n \to k$, there is an m-dense(Γ) set $Y \subseteq Z$ such that $\Psi(f, Y)$ holds, and,
  - for any partition $Z_0 \sqcup \cdots \sqcup Z_{\ell-1} = Z$ such that $\ell \leq Z_0 < \cdots < Z_{\ell-1}$, one of Z_i's is m-dense(Γ).

Note that "Z is m-dense(Γ)" can be expressed by a Δ_0-formula.

Put $Y_\Gamma(F) := \max\{m \mid F \text{ is } m\text{-dense}(\Gamma)\}$.

Theorem

Y_Γ is a set indicator for $\mathrm{WKL}_0 + \Gamma$.

Basic properties of indicators (review)

Theorem

If Y is an indicator for a theory T, then for any $n \in \omega$,

$$T \vdash \forall x \exists y Y(x, y) \geq n.$$

Theorem

If Y is an indicator for a theory T, then, T is a Π^0_2-conservative extension of $\mathrm{EFA} + \{\forall x \exists y Y(x, y) \geq n \mid n \in \omega\}$.

Let $F^Y_n(x) = \min\{y \mid Y(x, y) \geq n\}$.

Theorem

If Y is an indicator for a theory T and $T \vdash \forall x \exists y \theta(x, y)$ for some Σ_1-formula θ, then, there exists $n \in \omega$ such that

$$T \vdash \forall x \exists y < F^Y_n(x) \theta(x, y).$$

Basic properties of set indicators

Theorem

If Y is a set indicator for a theory T, then for any $n \in \omega$,

$$T \vdash \forall X \subseteq_{\inf} \mathbb{N}\, \exists F \subseteq_{\text{fin}} X (Y(F) \geq n).$$

Theorem

If Y is a set indicator for a theory T, then, T is a $\tilde{\Pi}^0_3$-conservative extension of $\mathrm{RCA}^*_0 + \{\forall X \subseteq_{\inf} \mathbb{N}\, \exists F \subseteq_{\text{fin}} X (Y(F) \geq n) \mid n \in \omega\}$.

Theorem

If Y is a set indicator for a theory T and $T \vdash \forall X \subseteq_{\inf} \mathbb{N}\, \exists F \subseteq_{\text{fin}} X \theta(F)$ for some Σ_1-formula θ, then, there exists $n \in \omega$ such that

$$T \vdash \forall Z \subseteq_{\text{fin}} \mathbb{N} (Y(Z) \geq n \rightarrow \exists F \subseteq Z\, \theta(F)).$$

Some consequences ($\tilde{\Pi}^0_3$-part of RT^2_2...)

- $\mathrm{WKL}_0 + \mathrm{RT}^2_2$ is a $\tilde{\Pi}^0_3$-conservative extension of $\mathrm{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X(F \text{ is } n\text{-dense}(\mathrm{RT}^2_2)) \mid n \in \omega\}$.
  $(\equiv \mathrm{RCA}_0 + \{n\mathrm{PH}^2_2 \mid n \in \omega\})$
- $\mathrm{WKL}_0 + \mathrm{RT}^2$ is a $\tilde{\Pi}^0_3$-conservative extension of $\mathrm{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X(F \text{ is } n\text{-dense}(\mathrm{RT}^2)) \mid n \in \omega\}$.
- $\mathrm{ACA}_0 + \mathrm{RT} = \mathrm{ACA}'_0$ is a Π^1_1-conservative extension of $\mathrm{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X(F \text{ is } n\text{-dense}(\mathrm{RT})) \mid n \in \omega\}$.
- $\mathrm{ACA}_0 + \mathrm{HT}(k)$ is a Π^1_1-conservative extension of $\mathrm{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X(F \text{ is } n\text{-dense}(\mathrm{HT}(k))) \mid n \in \omega\}$.
- $\mathrm{ACA}_0 + \mathrm{HT}$ is a Π^1_1-conservative extension of $\mathrm{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X(F \text{ is } n\text{-dense}(\mathrm{HT})) \mid n \in \omega\}$.
- ...

Here, HT denotes Hindman's theorem.

Some consequences (Generalized Parsons theorem)

Since ω^n-largeness implies n-density$(0 = 0)$, *i.e.*, a density notion for WKL_0, we have the following.

Theorem (Generalized Parsons theorem)

Let $\psi(F)$ be a Σ^0_1-formula with exactly the displayed free variables. Assume that for a given Ramsey-like statement Γ,

$$\mathsf{WKL}_0 + \Gamma \vdash \forall X \subseteq \mathbb{N}(X \text{ is infinite} \to \exists F \subseteq_{\mathrm{fin}} X\psi(F)).$$

Then, there exists $n \in \omega$ such that

$$\mathsf{WKL}_0 + \Gamma \vdash \forall Z \subseteq_{\mathrm{fin}} \mathbb{N}(Z \text{ is } n\text{-dense}(\Gamma) \to \exists F \subseteq Z\psi(F)).$$

In particular,

$$\mathsf{WKL}_0 \vdash \forall X \subseteq \mathbb{N}(X \text{ is infinite} \to \exists F \subseteq_{\mathrm{fin}} X\psi(F)).$$

Then, there exists $n \in \omega$ such that

$$\mathsf{WKL}_0 \vdash \forall Z \subseteq_{\mathrm{fin}} \mathbb{N}(Z \text{ is } \omega^n\text{-large} \to \exists F \subseteq Z\psi(F)).$$

Density with the base ACA_0

Definition (EFA, Density notion with the base ACA_0)

Given a Ramsey-like formula

$$\Gamma = (\forall f : [\mathbb{N}]^n \to k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y)),$$

- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be 0-*dense′*(Γ) if $|Z| > 4, \min Z > 2$,
- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be $(m+1)$-*dense′*(Γ) if
  - (for any $n, k < \min Z$ and) for any $f : [[0, \max Z]]^n \to k$, there is an m-dense′(Γ) set $Y \subseteq Z$ such that $\Psi(f, Y)$ holds, and,
  - for any partition $f : [Z]^3 \to \ell$ such that $\ell < \min Z$ there is an m-dense′(Γ) set $Y \subseteq Z$ which is f-homogeneous.

Put $Y'_\Gamma(F) := \max\{m \mid F \text{ is } m\text{-dense'}(\Gamma)\}$.

Theorem

Y'_Γ is a set indicator for $\mathrm{ACA}_0 + \Gamma$.

With ACA_0, one can always characterize the Π^1_1-part of Γ.

Density with the base WKL_0^*

Definition (EFA, Density notion with the base WKL_0^*)

Given a Ramsey-like formula

$$\Gamma = (\forall f : [\mathbb{N}]^n \to k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y)),$$

- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be 0-*dense*$^*(\Gamma)$ if $Z \neq \emptyset$,
- $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be $(m+1)$-*dense*$^*(\Gamma)$ if
  - (for any $n, k < \min Z$ and) for any $f : [[0, \max Z]]^n \to k$, there is an m-dense$^*(\Gamma)$ set $Y \subseteq Z$ such that $\Psi(f, Y)$ holds, and,
  - $Z \setminus [0, \exp(\min Z)]$ is m-dense$^*(\Gamma)$.

Put $Y_\Gamma^*(F) := \max\{m \mid F \text{ is } m\text{-dense}^*(\Gamma)\}$.

Theorem

Y_Γ^* is a set indicator for $\mathrm{WKL}_0^* + \Gamma$.

Conservation theorems for RT^n_k and $\mathrm{HT}(k)$ over WKL^*_0

- $\mathsf{WKL}^*_0 + \mathrm{RT}^n_k$ is a $\tilde{\Pi}^0_3$-conservative extension of $\mathsf{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X (F \text{ is } n\text{-dense}^*(\mathrm{RT}^n_k)) \mid n \in \omega\}$.
  $= \mathsf{RCA}^*_0$
- $\mathsf{WKL}^*_0 + \mathrm{RT} = \mathsf{ACA}'_0$ is a $\tilde{\Pi}^0_3$-conservative extension of $\mathsf{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X (F \text{ is } n\text{-dense}^*(\mathrm{RT})) \mid n \in \omega\}$.
- $\mathsf{WKL}^*_0 + \mathrm{HT}(k)$ is a $\tilde{\Pi}^0_3$-conservative extension of $\mathsf{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X (F \text{ is } n\text{-dense}^*(\mathrm{HT}(k))) \mid n \in \omega\}$.
  $= \mathsf{RCA}^*_0$
- $\mathsf{WKL}^*_0 + \mathrm{HT} = \mathsf{ACA}_0 + \mathrm{HT}$ is a $\tilde{\Pi}^0_3$-conservative extension of $\mathsf{RCA}^*_0 + \{\forall X \subseteq_{\mathrm{inf}} \mathbb{N}\, \exists F \subseteq_{\mathrm{fin}} X (F \text{ is } n\text{-dense}^*(\mathrm{HT})) \mid n \in \omega\}$.
- ...

Thus, $\mathsf{WKL}^*_0 + \mathrm{RT}^n_k$ and $\mathsf{WKL}^*_0 + \mathrm{HT}(k)$ are very weak, while $\mathsf{WKL}^*_0 + \mathrm{RT}$ and $\mathsf{WKL}^*_0 + \mathrm{HT}$ are not.

Thank you!

- Ludovic Patey and Y, The proof-theoretic strength of Ramsey's theorem for pairs and two colors, draft, available at http://arxiv.org/abs/1601.00050
- Y, On the strength of Ramsey's theorem without Σ_1-induction. Math. Log. Q., 59(1-2):108–111, 2013.