Weak Choice Principles in the Weihrauch Degrees

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Dagstuhl Problems (Sep. 2015)

1. (Pauly 2012) $(\exists k \in \omega)$ $\mathbf{AoUC} \star \mathbf{AoUC} \leq_W \mathbf{AoUC}^k$?
   Here, **AoUC** is the *all-or-unique* choice principle.
2. (Le Roux-Pauly 2015) $(\exists k \in \omega)$ $\mathbf{XC} \star \mathbf{XC} \leq_W \mathbf{XC}^k$?
   Here, **XC** is the *convex* choice principle.

It is easy to see that $\mathbf{LLPO} <_W \mathbf{AoUC} <_W \mathbf{XC} <_W \mathbf{WKL}$ (any recursion theorist can separate them).

Main Theorem (K. and Pauly)

1. Problem 1 is false: $\mathbf{LLPO} \star \mathbf{AoUC} \not\leq_W \mathbf{AoUC}^k$ for all k.
2. Problem 2 is false: $\mathbf{XC} \star \mathbf{AoUC} \not\leq_W \mathbf{XC}_k$ for all k.
   Here, $\mathbf{XC}_k$ is the k*-dimensional* convex choice principle.
3. However, it is true that

$$\mathbf{AoUC} \star \mathbf{AoUC} \star \mathbf{AoUC} \leq_W \mathbf{AoUC}^4 \star \mathbf{AoUC}^3.$$

A Π_2-principle is *non-uniformly computable* if any x-computable instance has an x-computable solution.

Equivalently, it has a σ*-computable* realizer, where f is σ*-computable* if it is decomposable into countably many computable functions.

(this is an effective version of σ*-continuity* in the sense of Nikolai Luzin).

Non-uniformly Computable Principles (below **WKL**)

1. **LLPO**: de Morgan's law for Σ^0_1-formulas.
2. $\mathbf{C}_n$: Given nonempty closed $F \subseteq \{1, \dots, n\}$, choose $i \in F$.
3. $\mathbf{C}_{[0,1],\#\leq n}$: Given nonempty closed $F \subseteq [0,1]$, if F has at most n many elements, choose $x \in F$.
4. **AoUC**: Given nonempty closed $F \subseteq [0,1]$, if $F = [0,1]$ or F is singleton, choose $x \in F$.
5. **XC**: Given nonempty convex closed $F \subseteq [0,1]$, choose $x \in F$.

Clear: $\mathbf{LLPO} \equiv_W \mathbf{C}_2 <_W \mathbf{C}_{[0,1],\#\leq 2} <_W \mathbf{AoUC} <_W \mathbf{XC} <_W \mathbf{WKL}$.

Definition (Weihrauch Reducibility)

$f \leq_W g$ iff there are computable H, K such that
for any realizer G of g, $K(\mathrm{id}, GH)$ realizes f.

(Brattka-Gherardi-Marcone) Classify $\mathbf{\Pi_2}$-theorems in the Weihrauch lattice.

There are some challenges to connect the Weihrauch lattice with intuitionistic linear logic:

- Yoshimura (submitted in 2013; still unpublished?):
  Some partial result using fibration in categorical logic.
- Kuyper: Some relationship with $\mathbf{EL_0}$ plus Markov's principle ($\mathbf{\Sigma^0_1}$-**DNE**) via realizability.

$\mathbf{EL_0}$ = Heyting Arithmetic **HA** restricted to quantifier-free induction **QF**-**IND**
with the axiom λ-convesion, the axiom of recursor,
and the quantifier-free axiom of choice **QF**-$\mathbf{AC_{00}}$

Note that $\mathbf{RCA_0} = \mathbf{EL_0} +$ "the law of excluded middle".

Constructive Reverse Mathematics

1. $\mathbf{EL_0}$ proves the equivalence of the following:
   - **BE**: every real number has a *binary expansion*.
     (a real number is represented by a rapid Cauchy sequence)
   - $\mathbf{C_{[0,1],\#\leq 2}}$: for any infinite binary tree T, if every level of T has at most 2 nodes, then T has an infinite path.
2. $\mathbf{EL_0}$ proves the equivalence of the following:
   - **IVT**: the *intermediate value theorem*.
   - **XC**: every infinite binary *convex* tree has an infinite path.
3. (Pauly 2010; Brattka-Gherardi-Hölzl 2015) $\mathbf{NASH} \equiv_W \mathbf{AoUC}^*$:
   Does $\mathbf{EL_0}$ ($+\mathbf{MP}$) prove the equivalence of the following?
   - **NASH**: every bi-matrix game has a *Nash equilibrium*.
   - **AoUC**: every infinite binary *all-or-unique* tree has an infinite path.

(1) and (2) are confirmed by Berger-Ishihara-K.-Nemoto
(we need some nontrivial work on eliminating Markov's principle).

There are a huge number of works in constructive reverse math...

Definition

For $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$,

1. $f \times g(x, z) = f(x) \times g(z)$.
2. $f \circ g(x) = \bigcup\{f(y) : y \in g(x)\}$.
3. $f \star g = \max_{\leq_W}\{f_0 \circ g_0 : f_0 \leq_W f \text{ and } g_0 \leq_W g\}$.

For $X = \mathbb{N}, 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}, \mathbb{R}$, etc., we have:

1. $\mathsf{C}_X \star \mathsf{C}_X \equiv_W \mathsf{C}_X \times \mathsf{C}_X \equiv_W \mathsf{C}_X$.
2. $\mathsf{PC}_X \star \mathsf{PC}_X \equiv_W \mathsf{PC}_X \times \mathsf{PC}_X \equiv_W \mathsf{PC}_X$.
3. $\mathsf{C}_{X,\#\leq n} \star \mathsf{C}_{X,\#\leq n} \equiv_W \mathsf{C}_{X,\#\leq n} \times \mathsf{C}_{X,\#\leq n}$.

(Brattka-Le Roux-Pauly) $\mathsf{XC} <_W \mathsf{XC} \times \mathsf{XC}$.

Dagstuhl Problems (Sep. 2015)

1. (Pauly 2012) $(\exists k \in \omega)\ \mathsf{AoUC} \star \mathsf{AoUC} \leq_W \mathsf{AoUC}^k$?
2. (Le Roux-Pauly 2015) $(\exists k \in \omega)\ \mathsf{XC} \star \mathsf{XC} \leq_W \mathsf{XC}^k$?

Main Theorem (K. and Pauly)

1. Problem 1 is false: $\mathbf{LLPO} \star \mathbf{AoUC} \not\leq_W \mathbf{AoUC}^k$ for all k.
2. Problem 2 is false: $\mathbf{XC} \star \mathbf{AoUC} \not\leq_W \mathbf{XC}_k$ for all k.
   Here, $\mathbf{XC}_k$ is the k-*dimensional* convex choice principle.
3. However, it is true that

$$\mathbf{AoUC} \star \mathbf{AoUC} \star \mathbf{AoUC} \leq_W \mathbf{AoUC}^4 \star \mathbf{AoUC}^3.$$

   In particular, we have

$$\mathbf{NASH} <_W \mathbf{NASH} \star \mathbf{NASH} \equiv_W \mathbf{NASH} \star \mathbf{NASH} \star \mathbf{NASH}.$$

$\mathsf{XC} \star \mathsf{AoUC} \not\leq_W \mathsf{XC}_k$ for all k.

(P_e, φ_e, ψ_e): the e-th triple constructed by the opponent **Opp**

- The e-th co-c.e. closed subset of $P_e \subseteq [0,1]^k$.
- The e-th partial computable $\varphi_e :\subseteq \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$.
- The e-th partial computable $\psi_e :\subseteq \mathbb{N}^{\mathbb{N}} \to [0,1]$.

The W-reduction proceeds as follows:

- We first give an all-or-unique tree $T_r \subseteq 2^{<\omega}$ and a map $J_r : 2^{\omega} \to \{\text{nonempty intervals}\}$.
- **Opp** reacts with a convex $P_r \subseteq [0,1]^k$, and ensure that
    - if z is a name of a point in P_r,
    - then $\varphi_r(z) = x$ is a path through T_r,
    - and $\psi_r(z)$ chooses an element of the interval $J_r(x)$,

  where Opp can use information on (names of) T_r and J_r to construct φ_r and ψ_r.
- Our purpose is to prevent **Opp**'s strategy.

By the recursion theorem, I know who I am.

- The e-th strategy constructs an a.o.u. tree T_e and an interval-valued map J_e.
- The q-th substrategy $\mathcal{S}_q$:
  - $\mathcal{S}_q$ acts under the assumption that *the substrategies $(\mathcal{S}_p)_{p<q}$ will eventually force the* **Opp***'s convex set $\mathbf{P_e}$ to be at most $(k-q)$-dimensional*.
  - The t-th action of $\mathcal{S}_q$ *forces the measure $\lambda^{k-q}(\tilde{P}_e)$ of a nonempty open subset $\tilde{P}_e$ of P_e to be less than or equal to $2^{q-t} \cdot \varepsilon_t$*, where $\varepsilon_t = \sum_{j=0}^{t+1} 2^{-j} < 2$.
  - If $\mathcal{S}_q$ acts infinitely often, then it *forces P_e to be at most $(k-q-1)$-dimensional* (under the assumption that P_e is convex).

$\mathsf{XC} \star \mathsf{AoUC} \not\leq_W \mathsf{XC}_k$ for all k.

How can we approximate the value of λ^{k-q} by an effective way?
Obvious obstacles:

- Even if we know that a co-c.e. closed $X \subseteq [0,1]^k$ is at most d-dimensional for some $d < k$, it is still possible that $X[s]$ can always be at least k-dimensional for all $s \in \omega$.

Fortunately, however, if a convex closed set $X \subseteq [0,1]^k$ is at most d-dimensional for some $d < k$:

- By convexity, X is a subset of d-dim. hyperplane L.
- By compactness, $X[s]$ for sufficiently large s is eventually covered by a *thin* k-parallelotope $\widehat{L}$ obtained by expanding d-hyperplane L.
- For instance, if $X \subseteq [0,1]^3$ is included in the plane $L = \{1/2\} \times [0,1]^2$, then for all $t \in \omega$, there is $s \in \omega$ such that $X[s] \subseteq \widehat{L}(2^{-t}) := [1/2 - 2^{-t}, 1/2 + 2^{-t}] \times [0,1]^2$ by compactness.
- We call such $\widehat{L}(2^{-t})$ as the *2^{-t}-thin expansion* of L.

The d-dim. measure λ^d is defined on Borel subsets of d-hyperplanes in $[0,1]^k$ whose values are consistent with the d-dim. volume (defined by the wedge product) on d-parallelotopes in $[0,1]^k$.

- Assume: We know that a convex set $X \subseteq [0,1]^k$ is at most d-dim., and moreover, a co-c.e. closed $\tilde{X} \subseteq X$ satisfies that $\lambda^d(\tilde{X}) < r$.
- Given $\varepsilon > 0$, there must be a rational closed subset Y of a d-hyperplane L such that $\tilde{X}$ is covered by the *ε-thin expansion* $\widehat{Y}(\varepsilon)$ of Y, and moreover, Y is *very close* to $\tilde{X}$.
  - If Y is a rational closed subset of a d-hyperplane, one can calculate $\lambda^d(Y)$.
  - Indeed, we can compute the maximum value $m^d(Y,\varepsilon)$ of $\lambda^d(\widehat{Y}(\varepsilon) \cap L')$ where L' ranges over all d-hyperplanes.
  - For instance, if $Y = [0,s] \times \{y\}$, it is easy to see that
$$m^1(Y,\varepsilon) = \sqrt{s^2 + (2\varepsilon)^2}.$$
- If $\lambda^d(\tilde{X}) < r$, given n, one can effectively find s, Y, ε such that
$$\tilde{X}[s] \subseteq \widehat{Y}(\varepsilon) \text{ and } m^d(Y,\varepsilon) < r + 2^{-n}.$$
- In this way, if the inequality $\lambda^d(\tilde{X}) < r$ holds for a co-c.e. closed subset $\tilde{X}$ of a d-dimensional convex set X, then one can effectively confirm this fact.

$\mathsf{XC} \star \mathsf{AoUC} \not\leq_W \mathsf{XC}_k$ for all k.

Opp: (convex) closed $P_e \subseteq [0,1]^k$, which helps φ_e to find a path p of T_e, and ψ_e to find an element of $J_e(p)$.

The action of the q-th substrategy $\mathcal{S}_q$:

1. Ask whether $\varphi_e(z)$ already computes a node of length at least $p+1$ for any name z of an element of P_e.
   - Yes $\Rightarrow$ Go next // No $\Rightarrow$ Wait.
2. Ask whether there is some $\tau \in 2^{q+1}$ such that any point in P_e has a name z such that $\varphi_e(z)$ does not extend τ.
   - No $\Rightarrow$ Go next.
   - Yes $\Rightarrow$ Let T_e be a tree with a *unique* path $\tau^\frown 0^\omega$; then we win.
3. Now *$\mathcal{S}_q$ believes that $(\mathcal{S}_p)_{p<q}$ eventually forces P_e to be at most $(k-q)$-dimensional*. Under this assumption, *$\mathcal{S}_q$ believes that $\mathcal{S}_q$ has forced $\lambda^{k-q}(\tilde{P}_e) < 2^{q-t+1} \cdot \varepsilon_{t-1}$* ($\tilde{P}_e$ is an open subset of P_e) by $\mathcal{S}_q$'s $(t-1)$-st action.
4. Ask whether for any name z of a point of P_e, whenever $\varphi_e(z)$ extends 0^q1, the value of $\psi_e(z)$ is already approximated with precision 3^{-t-2}.
   - Yes $\Rightarrow$ Go next // No $\Rightarrow$ Wait.

The action of the q-th substrategy $\mathcal{S}_q$ (Continued):

- We have a nonempty interval $J_e(0^q1)$ at the current stage.
- I_0, I_1: sufficiently separated subintervals of $J_e(0^q1)$.
- V: names of points in P_e whose φ_e-values extend 0^q1.
- $\mathcal{S}_q$ believes that $\mathcal{S}_q$ has already forced $\lambda^{k-q}(\delta[V]) \leq 2^{q-t+1} \cdot \varepsilon_{t-1}$, where δ is an open representation of $[0,1]^k$.
- Q_i: the set of all points in $\overline{\delta[V]}$ all of whose names are still possible to have ψ_e-values in I_i with precision 3^{-t-2}. One can show:
  - Q_i is effectively compact.
  - $\lambda^{k-q}(Q_0 \cap Q_1) = 0$ whenever P_e is at most $(k-q)$-dim.
  - Therefore, $\lambda^{k-q}(Q_i) \leq 2^{q-t} \cdot \varepsilon_{t-1}$ for some $i < 2$.
- Finally, ask whether there is a witness for the above. That is, ask whether one can find s, Y, ε, i such that

$$Q_i[s] \subseteq \widehat{Y}(\varepsilon) \text{ and } m^{k-q}(Y, \varepsilon) < 2^{q-t} \cdot \varepsilon_t.$$

  - No $\Rightarrow$ Wait.
  - Yes $\Rightarrow$ Put $J_e(0^q1) = I_i$ and go to the next action $t+1$.

- The previous action of $\mathcal{S}_q$ forces that $\delta[V] \subseteq Q_i$; therefore, $\lambda^{k-q}(\delta[V]) \leq \lambda^{k-q}(Q_i) \leq 2^{q-t} \cdot \varepsilon_t$.
- If $\mathcal{S}_q$ acts infinitely often, then this forces $\lambda^{k-q}(\delta[V]) = 0$; therefore convexity of P_e implies $\lambda^{k-q}(P_e) = 0$ since $\delta[V]$ is an open subset of P_e.

Given (T_r, J_r), **Opp** reacts with $(P_{f(e)}, \varphi_{f(e)}, \psi_{f(e)})$.
By the Rec. Thm., there is r s.t. $(P_r, \varphi_r, \psi_r) = (P_{f(r)}, \varphi_{f(r)}, \psi_{f(r)})$.
Suppose: **Opp** wins with this triple (P_r, φ_r, ψ_r)

- Then $\mathcal{S}_q$ eventually forces P_r to be $(k - q - 1)$-dimensional; therefore, $(\mathcal{S}_q)_{q<k}$ forces P_r to be zero-dimensional.
- Since P_r is convex (if **Opp** wins), P_r is a singleton or empty.
- Then, there is some $\tau \in 2^{q+1}$ such that any point in P_r has a name z such that $\varphi_r(z)$ does not extend τ.
- Then $\mathcal{S}_q$ ensures that T_r has a *unique* path $\tau^\frown 0^\omega$.
- Thus, φ_r fails to choose a path of T_r; hence **Opp** loses.

LLPO $\star$ AoUC $\not\leq_W$ AoUC^k for all k.

Proof of **LLPO** $\star$ **AoUC** $\not\leq_W$ **AoUC***:
Easy. Use the similar argument □

$\mathsf{NASH} <_W \mathsf{NASH} \star \mathsf{NASH} \equiv_W \mathsf{NASH} \star \mathsf{NASH} \star \mathsf{NASH}$.

Lemma

1. $\mathsf{AoUC}^m \star \mathsf{AoUC}^k \leq_W \mathsf{C}_{2^k} \star (\mathsf{AoUC}^{m \cdot 2^k + k})$.

   In particular, $\mathsf{AoUC} \star \mathsf{AoUC} \leq_W \mathsf{LLPO} \star (\mathsf{AoUC}^3)$.

2. $\mathsf{AoUC}^l \star \mathsf{C}_m \leq_W \mathsf{AoUC}^{l \cdot m} \times \mathsf{C}_m$.

Corollary

$$\mathsf{AoUC}^l \star \mathsf{AoUC}^m \star \mathsf{AoUC}^k \leq_W \mathsf{AoUC}^{(l+1) \cdot 2^k} \star \mathsf{AoUC}^{m \cdot 2^k + k}.$$

In particular,

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