Characterizing uniform provability by intuitionistic provability

Makoto Fujiwara

School of Information Science,
Japan Advanced Institute of Science and Technology (JAIST)

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Conceptual Interest

Two Kinds of Constructive Provability

For existence statement $\forall x(\varphi(x) \to \exists y \psi(x, y))$,

1. its provability in **computable analysis** or **computational complexity theory** etc. means that there is an algorithm to output y from x.

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   there is an algorithm to output y from x and it is verified mathematically that the algorithm works.

2. its provability in **constructive mathematics** roughly means (by realizability interpretation) that

   there is an algorithm to output y from x and there is also another algorithm to verify that the algorithm works.

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How is the gap between these two kinds of provability?

Constructive Mathematics (Early 20th Century –)

Constructive mathematics (Brouwer, Markov, Bishop etc.) is distinguished from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase "there exists" as "we can construct".*

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In order to work constructively, we need to re-interpret not only the existential quantifier but all the logical connectives and quantifiers as instructions on how to construct a proof of the statement involving these logical expressions (BHK-interpretation).

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Formal Systems

	Intuitionistic Logic	Classical Logic
	HA	PA $(= \mathsf{HA} + \mathrm{LEM})$
Two-sorted	EL	RCA $(= \mathsf{EL} + \mathrm{LEM})$

- LEM denotes the law-of-excluded-middle axiom $A \vee \neg A$.
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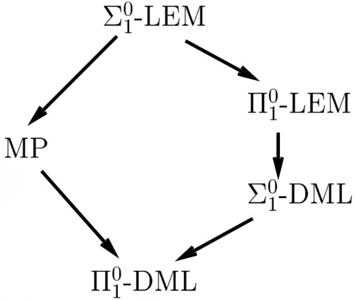
Remark. EL (or EL_0) serves as base theory for **Constructive Reverse Mathematics** (Ishihara, Nemoto etc., 2000's–).

Arithmetical Hierarchy of Logical Principles (Ishihara 1993, Akama et al. 2004)



- MP : $\forall\alpha(\neg\neg\exists x(\alpha(x)=0) \to \exists x(\alpha(x)=0))$
- Σ^0_1-LEM : $\forall\alpha(\exists x(\alpha(x)=0) \vee \neg\exists x(\alpha(x)=0))$
- Σ^0_1-DML :
$$\forall\alpha,\beta\left(\begin{array}{l}\neg(\exists x(\alpha(x)=0) \wedge \exists x(\beta(x)=0)) \\ \to (\neg\exists x(\alpha(x)=0) \vee \neg\exists x(\beta(x)=0))\end{array}\right)$$

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To reveal the non-uniformity, the following sequential version (Hirst, Mummert etc., 2000's–) has been investigated.

$$\forall \langle f_n \rangle_{n \in \mathbb{N}} \left(\forall n \varphi(f_n) \to \exists \langle g_n \rangle_{n \in \mathbb{N}} \forall n \psi(f_n, g_n) \right).$$

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Examples.

	Pointwise	Sequential
JD (The existence of Jordan decomposition for real square matrices)	RCA	ACA
RT^1 (Infinite pigeonhole principle)	RCA	ACA
IVT (Intermediate value theorem)	RCA	WKL
TET (Tietze extension theorem)	RCA	RCA
EMT (Effective marriage theorem)	RCA	RCA

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- $\mathrm{DIC} : \forall \alpha \in \mathbb{R} \, (\alpha \leq 0 \vee \alpha \geq 0)$.

Fact.

Over EL,

- $\mathrm{TRIC} \leftrightarrow \Sigma^0_1\text{-LEM}$.
- $\mathrm{DIC} \leftrightarrow \Sigma^0_1\text{-DML}$.

Fact.

Over RCA,

- $\mathrm{Seq(TRIC)} \leftrightarrow \mathrm{ACA}$.
- $\mathrm{Seq(DIC)} \leftrightarrow \mathrm{WKL}$.

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Fact.

Over RCA,

- $\mathrm{Seq}(\mathrm{TRIC}) \leftrightarrow \mathrm{ACA}$.
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Proposition. (Ishihara 2005)

- $\mathrm{EL} \vdash \mathrm{ACA} \leftrightarrow \Sigma^0_1\text{-}\mathrm{LEM} + \Pi^0_1\text{-}\mathrm{AC}_{00}$.
- $\mathrm{EL} \vdash \mathrm{WKL} \leftrightarrow \Sigma^0_1\text{-}\mathrm{DML} + \Pi^0_1\text{-}\mathrm{AC}^{\lor}_0$.

[Intuitionistic ⇒ Sequential]

	Second-order	Higher-order
$\mathsf{RCA} + \mathrm{WKL}$	Dorais 2014	Kohlenbach/F. 2015
RCA	Dorais 2014	Hirst/Mummert 2011

[Intuitionistic $\Rightarrow$ Sequential]

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Question.

How about the converse direction?

[Intuitionistic $\Rightarrow$ Sequential]

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To answer this question, we formalize uniform provability in the exact way rather than sequentialization.

Formalization of Uniform Provability

Finite-type Systems

- Hilbert-type system E-HA$^\omega$ (resp. E-PA$^\omega$) is the finite type extension of HA (resp. PA).
- E-PA$^\omega$:= E-HA$^\omega$ + **LEM**$(\mathrm{A} \vee \neg \mathrm{A})$.
- RCA$^\omega$:= E-PA$^\omega$ + $\mathrm{QF\text{-}AC}^{1,0}$.

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Fact.

- RCA$^\omega$ is a conservative extension of RCA.
- WKL$^\omega$(:= RCA$^\omega$ + WKL) is a conservative extension of RCA + WKL.
- ACA$^\omega$(:= RCA$^\omega$ + ACA) is a conservative extension of RCA + ACA.

Uniform provability in $\boldsymbol{\Gamma}$:

1. There exists a (Gödel prim. rec.) term $t^{1\to 1}$ of RCA^ω s.t.

$$\boldsymbol{\Gamma} \vdash \forall f\,(\varphi(f) \to \psi(f, tf))\,.$$

2. There exists a (Kleene prim. rec.) term t^1 of RCA s.t.

$$\boldsymbol{\Gamma} \vdash \forall f\,(\varphi(f) \to t|f\downarrow \wedge \psi(f, t|f))\,,$$

where

$$\alpha(\beta) := \begin{cases} \alpha(\bar{\beta}n) - 1 \text{ where n is the least } n' \text{ s.t. } \alpha(\bar{\beta}n') \neq 0. \\ \uparrow \text{ if there is no such } n'. \end{cases}$$

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Remark. RCA $\vdash \mathrm{Seq}(\mathrm{S})$ follows from the fact that S is uniformly provable in RCA.

Proposition 1. (F. 2015)

If there exists a term t^1 of RCA such that

$$\mathsf{RCA} \vdash \forall f\,(\varphi(f) \to t|f\downarrow \wedge \psi(f, t|f)),$$

then

$$\mathsf{EL} + \mathrm{MP} \vdash \forall f\,(\varphi(f) \to \exists g\psi(f, g)),$$

provided that $\varphi(f) \in \mathbf{J}$ and $\psi(f, g)$ is equivalent to some formula $\forall w^\rho \exists s^0 \psi_{qf}(f, g, w, s)$ over $\mathsf{EL} + \mathrm{MP}$ $(\rho \in \{0, 1\})$.

- $\mathbf{J}$ is the class of formulas defined inductively as;
  - A_{qf} is in $\mathbf{J}$.
  - If A_1, A_2 are in $\mathbf{J}$, then $A_1 \wedge A_2$, $A_1 \vee A_2$, $\forall u^\rho A_1$ and $\exists v^\rho A_1$ are in $\mathbf{J}$, where $\rho \in \{0, 1\}$.
  - If A is in $\mathbf{J}$, then $\forall u^\rho \exists v^0 A_{qf} \to A$ is in $\mathbf{J}$, where $\rho \in \{0, 1\}$.

Key Lemma. (Conservation Result)

For $\varphi(f) \in \mathbf{J}$,

$$\mathsf{RCA} \vdash \forall f \left(\varphi(f) \to t \mid f \downarrow \wedge \forall w^{\rho} \exists s^{0} \psi_{qf}(f, t \mid f, w, s)\right)$$

$$\Rightarrow \mathsf{EL} + \mathrm{MP} \vdash \forall f \left(\varphi(f) \to t \mid f \downarrow \wedge \forall w \exists s \psi_{qf}(f, t \mid f, w, s)\right).$$

Proof. By Kuroda's negative translation.

Corollary 1.

There exists a term t^1 of RCA such that

$$\mathrm{RCA} \vdash \forall f\,(\varphi(f) \to t|f\downarrow \wedge \psi(f, t|f))$$

if and only if

$$\mathrm{EL} + \mathrm{MP} \vdash \forall f\,(\varphi(f) \to \exists g \psi(f, g)),$$

provided that $\varphi(f) \in \mathbf{K}$ and $\psi(f, g)$ is equivalent to some formula $\forall w^{\rho} \exists s^0 \psi_{qf}(f, g, w, s)$ over $\mathrm{EL} + \mathrm{MP}$ ($\rho \in \{0, 1\}$).

- $\mathbf{K}$ is the class of formulas defined inductively as;
  - A_{qf} and $\exists v A_{qf}$ are in $\mathbf{K}$.
  - If A_1, A_2 are in $\mathbf{K}$, then $A_1 \wedge A_2$, $\forall u A_1$ are in $\mathbf{K}$.
  - If A is in $\mathbf{K}$, then $\forall u \exists v A_{qf} \to A$ is in $\mathbf{K}$.
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- **K** is the class of formulas defined inductively as;
  - A_{qf} and $\exists v A_{qf}$ are in **K**. $\Rightarrow \forall u \exists v \varphi_{qf}(f, u, v) \in \mathbf{K}$.
  - If A_1, A_2 are in **K**, then $A_1 \wedge A_2$, $\forall u A_1$ are in **K**.
  - If A is in **K**, then $\forall u \exists v A_{qf} \to A$ is in **K**.
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Proposition 2.

Let $\forall f\,(\varphi(f) \to \exists g \psi(f,g))$ be a $\mathcal{L}(\mathrm{EL})$-formula such that $\varphi(f)$ is equivalent to $\forall u^0 \exists v^0 \varphi_{qf}(f,u,v)$ and $\psi(f,g)$ is equivalent to $\forall w^\rho \exists s^\tau \psi_{qf}(f,g,w,s)$ over EL ($\rho, \tau \in \{0,1\}$).

1. If there exists a term t^1 of RCA such that

$$\mathrm{RCA}+\mathrm{WKL}+\mathrm{QF\text{-}AC}^{0,1} \vdash \forall f\,(\varphi(f) \to t|f \downarrow \wedge \psi(f, t|f)),$$

then

$$\mathrm{EL} \vdash \forall f\,(\varphi(f) \to \exists g \psi(f,g)).$$

2. If there exists a term $t^{1\to1}$ of RCA^ω such that

$$\mathrm{RCA}^\omega + \mathrm{WKL} + \mathrm{QF\text{-}AC}^{0,1} \vdash \forall f\,(\varphi(f) \to \psi(f, tf)),$$

then

$$\mathrm{EL} \vdash \forall f\,(\varphi(f) \to \exists g \psi(f,g)).$$

The main tool for the proof is so-called **monotone Dialectica interpretation** (Kohlenbach, 1990's), which is a combination of Gödel's Dialectica interpretation with Howard's majorizability construction.

Sketch of the Proof of 2. (That of 1 is similar.)

Let S denote $\forall f\,(\varphi(f) \to \exists g \psi(f,g))$.

1. Note that $\mathsf{RCA}^\omega + \mathrm{QF\text{-}AC}^{0,1} \vdash \mathrm{WKL} \to \bar{\mathrm{S}}[t]^-$, where $\bar{\mathrm{S}}[t]^- :=$ $\forall f^1, V^1\,(\forall u^0 \varphi_{qf}(f,u,Vu) \to \forall w \exists s \psi_{qf}(f, tf, w, s))$.

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2. By **elimination of extensionality** and **negative translation**, we have $\mathsf{WE\text{-}HA}^\omega + \mathrm{QF\text{-}AC} + \mathrm{M}^\omega \vdash \mathrm{WKL} \to \bar{\mathrm{S}}[t]^-$.

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3. By **monotone Dialectica interpretation**, we have $\mathrm{WE\text{-}HA}^\omega \vdash \bar{\mathrm{S}}[t]^-$.
4. Then $\mathrm{WE\text{-}HA}^\omega + \mathrm{QF\text{-}AC}^{0,0} \vdash \bar{\mathrm{S}} :=$ $\forall f^1\,(\forall u^0 \exists v^0 \varphi_{qf}(f,u,v) \to \exists g \forall w \exists s \psi_{qf}(f,g,w,s))$.

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5. By the conservation, we have $\mathrm{EL} \vdash \bar{\mathrm{S}}$, and hence $\mathrm{EL} \vdash \mathrm{S}$. □

Remark.

In fact, one can even add Σ^0_1-UB instead of WKL.

- Σ^0_1-UB is a slight extension of QF-FAN, which is classically false but consistent with RCA$^\omega$.
- RCA$^\omega + \Sigma^0_1$-UB $\vdash$ WKL. (Kohlenbach)

Corollary 2.

Let $\forall f\,(\varphi(f) \to \exists g \psi(f,g))$ be a $\mathcal{L}(\mathrm{EL})$-formula such that $\varphi(f)$ is equivalent to $\forall u^0 \varphi_{qf}(f,u)$ and $\psi(f,g)$ is equivalent to $\forall w^\rho \exists s^\tau \psi_{qf}(f,g,w,s)$ over EL ($\rho, \tau \in \{0,1\}$). TFAE.

1. $\mathrm{EL} \vdash \forall f\,(\varphi(f) \to \exists g \psi(f,g))$.
2. There exists a term t^1 of RCA such that $\mathrm{RCA} + \mathrm{WKL} + \mathrm{QF\text{-}AC}^{0,1} \vdash$ $\forall f\,(\varphi(f) \to t \mid f \downarrow \wedge\ \psi(f, t \mid f))$.
3. There exists a term $t^{1\to1}$ of RCA^ω such that $\mathrm{RCA}^\omega + \mathrm{WKL} + \mathrm{QF\text{-}AC}^{0,1} \vdash \forall f\,(\varphi(f) \to \psi(f, tf))$.

- $(1 \to 2)$ is as before.
- $(1 \to 3)$ is by modified realizability interpretation (Hirst and Mummert 2011).

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- $(1 \to 2)$ is as before.
- $(1 \to 3)$ is by modified realizability interpretation (Hirst and Mummert 2011). The syntactical restriction of $\varphi(f)$ is crucial for this part.

Application.

For example, Kierstead's effective marriage theorem EMT has the required syntactical form in Corollary 2 and uniformly provable in RCA, then it follows that EMT is provable in EL.

Remark.

- It is known that many existence theorems are formalized as a Π^1_2 formula of the syntactical form in Corollary 2 and most of practical existence theorems are formalized as that in Corollary 1.
- Analogous results for EL_0, RCA_0, RCA^ω_0 instead of EL, RCA, RCA^ω also hold.
- All of the proofs are syntactical.

Possible Directions

1. Investigation of uniform provability in stronger systems:
   - Is there natural existence theorem which is uniformly provable in $\mathsf{RCA} + \mathrm{ACA}$, but not uniformly provable in $\mathsf{RCA} + \mathrm{WKL}$?

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   - Is it possible to reduce uniform provability in RCA $+ \mathrm{WKL} + \mathrm{RT}^2_2$ to intuitionistic provability?

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1. Investigation of uniform provability in stronger systems:
   - Is there natural existence theorem which is uniformly provable in RCA $+ \mathrm{ACA}$, but not uniformly provable in RCA $+ \mathrm{WKL}$?
   - Is it possible to reduce uniform provability in RCA $+ \mathrm{WKL} + \mathrm{RT}^2_2$ to intuitionistic provability?
2. Formalize the relative uniform provability (like Weihrauch reducibility) and characterize it by (semi-)intuitionistic provability. (Rutger Kuyper, preprint, 2015)

References

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Thank you for your attention!