The downward directed grounds hypothesis

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Abstract

We show that if M and N are ground models of the universe V, that is,

- $M, N \subseteq V$ are transitive models of ZFC,
- $V = M[G] = N[H]$ for some generic filters $G \subseteq \mathbb{P} \in M$ and $H \subseteq \mathbb{Q} \in N$,

then M and N have a common ground model. We also show that the intersection of all ground models of V is a model of ZFC, and is a "core" of V in the sense of forcing.

Definability of ground models

Throughout this talk, forcing means set forcing.

Fact (Laver (2007), Woodin)

In the forcing extension $V[G]$ of V, the universe V is a (first order) definable class in $V[G]$ with some parameters from V.

In other words:
If $M \subseteq V$ is a ground model of V, then M is definable.
So every ground model of V is definable by some first order formula.

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So every ground model of V is definable by some first order formula.

Uniform definability of ground models

Actually all grounds model can be defined by some uniform way.

Fact (Fuchs-Hamkins-Reitz (2015))

There is a first order formula $\varphi(x, y)$ such that:

1. *For each set r, the class $W_r = \{x : \varphi(x, r)\}$ is a ground model of V, that is, it is a transitive model of ZFC, and there is a poset $\mathbb{P} \in W_r$ and a $(W_r, \mathbb{P})$-generic G with $V = W_r[G]$ ($V = W_r$ is possible).*
2. *For every transitive model $M \subseteq V$ of ZFC, if M is a ground model of V, then there is r with $M = W_r$.*

The mantle

This result allow us to study the structure of the collection of ground models $\{W_r : r \in V\}$ in ZFC:e.g.,

- One can define (in ZFC) the intersection of two ground models.
- One can ask (in ZFC) whether $\forall r \exists s\, (W_s \subsetneq W_r)$?

The "mantle" is a natural concept indicated by uniform definability.

Definition

The <u>mantle</u> $\mathbb{M}$ is the intersection of all ground models of V.

The mantle is a first order definable class $\{x : \forall r\, \varphi(x, r)\}$.

- There are many open questions about the mantle.

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- There are many open questions about the mantle.

An important question about the mantle is:

Question (Fuchs-Hamkins-Reitz)

Is the mantle a model of ZF or ZFC?

If V is $L[X]$, HOD, K, class forcing extensions of there models, or other known models, then the mantle is a model of ZFC.

DDG

Another interesting question is the downward directedness of the ground models:

Does every two grounds W_0, W_1 have a common ground $W \subseteq W_0, W_1$?

Definition (Fuchs-Hamkins-Reitz)

The downward directed grounds hypothesis (DDG, for short) is the assertion that every two ground models have a common ground model: $\forall r_0, r_1 \exists r\, (W_r \subseteq W_{r_0} \cap W_{r_1})$.

The strong downward directed grounds hypothesis (strong DDG, for short) is the assertion that every set X, the collection $\{W_r : r \in X\}$ of ground models have a common ground model: $\forall X \exists r\, \forall s \in X\, (W_r \subseteq W_s)$.

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Fact (Fuchs-Hamkins-Rietz)

1. *Many known models such as $L[X]$, HOD, K, class forcing extensions, $\ldots$ satisfy the strong DDG.*
2. *If the strong DDG holds, then the mantle is a model of ZFC.*

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Does the DDG always hold? How is the strong DDG?

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Question (Fuchs-Hamkins-Reitz)

Does the DDG always hold? How is the strong DDG?

Main theorem 1

Theorem

The strong DDG always holds. Consequently, the mantle is a model of ZFC.

Other consequences:bedrock

Definition

A bedrock is a minimal ground model of V. A solid bedrock is a minimum ground of V.

It was unknown whether every bedrock is solid.

Corollary

1. If a bedrock exists, then it is the solid bedrock.
2. V has a bedrock if and only if V has only set many ground models, that is, there is a set X such that $\forall r \exists s \in X \, (W_r = W_s)$.
3. V has a bedrock if and only if the mantle itself is a ground model of V.

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3. V has a bedrock if and only if the mantle itself is a ground model of V.

Other consequences: Invariant under forcing

The mantle is a "core" of V in the sense of forcing.

Corollary

The mantle is a parameter free definable model of ZFC, and is a largest forcing-invariant definable class:

1. For every generic extension $V[G]$ of V and ground model W of V, we have $\mathbb{M}^V = \mathbb{M}^{V[G]} = \mathbb{M}^W$.
2. If $\mathbb{N}$ is a definable class which is invariant under forcing, then $\mathbb{N} \subseteq \mathbb{M}$.

Other consequences: Generic multiverse

Definition (Woodin (2011))

A generic multiverse is a “minimum” collection $\mathcal{F}$ of (countable) models of ZFC such that $\mathcal{F}$ is closed under taking ground models and generic extensions:

1. $M \in \mathcal{F}$, $N \subseteq M$ is a ground model of $M \Rightarrow N \in \mathcal{F}$.
2. $M \in \mathcal{F}$, $N \supseteq M$ is a forcing extension of $M \Rightarrow N \in \mathcal{F}$.
3. $M, N \in \mathcal{F} \Rightarrow \exists U_0, \ldots, U_n \in \mathcal{F}$ such that $U_0 = M$, $U_n = N$, and U_{i+1} is a ground model or a forcing extension of U_i.

Corollary

1. Every two members of a generic multiverse have a common ground model:
   $M, N \in \mathcal{F} \Rightarrow \exists U \in \mathcal{F}$, U is a ground model of M and N.
2. For every M, $N \in \mathcal{F}$, we have $\mathbb{M}^M = \mathbb{M}^N$.
3. For some/any $N \in \mathcal{F}$, if $\mathbb{M}^N$ is a bedrock of N then $\mathbb{M}^N$ is the minimum element of $\mathcal{F}$, and every member of $\mathcal{F}$ is a forcing extension of $\mathbb{M}^N$.

The mantle is a "core" of generic multiverse.

Truth of multiverse

Definition (Woodin)

A multiverse truth is a sentence φ which is true in every member of generic multiverse.

- A multiverse truth is a sentence which is "absolutely true" in the sense of forcing.

Proposition (Woodin)

There is a computable translation $\varphi \mapsto \varphi^*$ on sentences such that for every $M \in \mathcal{F}$, a sentence φ is a multiverse truth if and only if φ^* holds in M.

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Corollary

1. Let $M \in \mathcal{F}$. Then a sentence φ is a multiverse truth if and only if for every ground model W of M, every forcing $\mathbb{P} \in W$ forces φ in W.
2. If $\mathbb{M}^N \in \mathcal{F}$ for some/any $N \in \mathcal{F}$, then φ is a multiverse truth if and only if every forcing $\mathbb{P} \in \mathbb{M}^N$ forces φ in $\mathbb{M}^N$.

Key of the proof

Definition

Let $M \subseteq V$ be a transitive model of ZFC. Let κ be a cardinal. M satisfies the κ-global covering property for V if for every ordinal α and every function $f : \alpha \to ON$, there is $F \in M$ such that $F : \alpha \to [On]^{<\kappa}$ and $f(\beta) \in F(\beta)$ for $\beta < \alpha$.

Fact (Bukovsky (1973))

Let $M \subseteq V$ be a transitive model of ZFC. Then the following are equivalent:

1. *M satisfies the κ-global covering property for V for some κ.*
2. *M is a ground model of V.*

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Rough sketch of the proof

Let $W_0, W_1 \subseteq V$ be ground models. Then there is some cardinal κ such that W_0 and W_1 satisfy the κ-global covering property for V.

For each sufficiently large $\theta > \kappa$, we can find a transitive model $M^\theta \subseteq \mathcal{H}_\theta$ of ZFC-P such that:

1. $\theta \subseteq M^\theta \in W_0 \cap W_1$.
2. M^θ satisfies the κ^+-global covering property for $\mathcal{H}_\theta$.
3. M^θ is unique in the following sense: For every N, if
    3.1 N is a transitive model of ZFC-P with $\theta \subseteq N \subseteq \mathcal{H}_\theta$,
    3.2 N satisfies the κ^+-global covering property for $\mathcal{H}_\theta$,
    3.3 $\mathcal{P}(\kappa^+) \cap N = \mathcal{P}(\kappa^+) \cap M$,
    Then $M = N$.

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Rough sketch of the proof:conti.

By the uniqueness of the M^θ's, we can find a sequence $\langle M^\theta : \theta > \kappa \rangle$ such that:

1. M^θ satisfies the κ^+-global covering property for $\mathcal{H}_\theta$.
2. If $\theta < \chi$, then $M^\theta = M^\chi \cap \mathcal{H}_\theta$.

Let $M = \bigcup_{\theta > \kappa} M^\theta$. One can check that M is a model of ZFC. Moreover, M satisfies the κ^+-global covering property for V. Now we have that M is a ground model of V by Bukovsky's theorem.

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Bedrock and large cardinals

Fact (Fuchs-Hamkins-Reitz)

It is consistent that V has a bedrock, moreover there is a class forcing $\mathbb{P} \subseteq V$ such that if G is $(V, \mathbb{P})$-generic, then $\mathbb{M}^{V[G]} = V[G]$.

This class forcing is a standard Easton support iteration of closed forcings, so it preserves almost all large cardinals.

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"V has a bedrock" is consistent with almost all large cardinals.

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No bedrock and large cardinals

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It is consistent that V has no bedrock, moreover there is a class forcing $\mathbb{P} \subseteq V$ such that if G is $(V, \mathbb{P})$-generic, then $V[G]$ has no bedrock (so $\mathbb{M}^{V[G]}$ is not a ground model of $V[G]$).

This poset $\mathbb{P}$ can preserve supercompact cardinals.

Fact

"No bedrock exist" is consistent with supercompact cardinals.

However it is unknown whether "no bedrock exist" is consistent with large cardinals which are stronger than supercompact cardinals.

We will show that some large cardinal is inconsistent with "no bedrock".

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Definition

An infinite cardinal κ is supercompact if for every cardinal $\lambda > \kappa$, there is an inner model M of ZFC and an elementary embedding $j : V \to M$ such that:

1. The critical point of j is κ.
2. $\lambda < j(\kappa)$.
3. M is closed under λ-sequences, that is, $[M]^\lambda \subseteq M$.

- Consistencies of many interesting (independent) propositions (e.g., Martin's Maximum) are implied by supercompact cardinals.
- If supercompact cardinal exists, then every projective set of reals is Lebesgue measurable, has a Baire property, has a perfect set property, etc.

New large cardinal

Definition

An infinite cardinal κ is hyper huge if for every cardinal $\lambda > \kappa$, there is an inner model M of ZFC and an elementary embedding $j : V \to M$ such that:

1. The critical point of j is κ.
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3. M is closed under $j(\lambda)$-sequences.

Lemma

Hyper huge cardinal is a supercompact cardinal, and a limit of supercompact cardinals.

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Main theorem 2

Theorem

Suppose hyper huge cardinal exists. Then V has only set many ground models.
Consequently, the mantle is a ground model of V, hence V has a solid bedrock.

- This means that if very large cardinal exists, then V must be very close to its "core".
- This also shows that there is some essential "gap" between supercompact cardinals and very large cardinals in the sense of forcing.

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Lemma

Suppose κ is hyper huge. If $W \subseteq V$ is a ground model, then there is a poset $\mathbb{P} \in W$ and a $(W, \mathbb{P})$-generic G such that

1. $|\mathbb{P}| < \kappa$.
2. $V = W[G]$.

Moreover, if κ is hyper huge, then V has only $< \kappa$ many ground models.

Applying the strong DDG, we can conclude that the mantle is a ground model of V.

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Thank you for your attention!