Isotrivial VMRT structures of complete intersection type

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X: uniruled smooth proj. var.

$\mathcal{K}$: a family of rat. curves on X, covering & minimal

$x \in X$ general

$\mathcal{K}_x$: curves in $\mathcal{K}$ thru x

$\mathcal{K}_x \overset{\tau_x}{\dashrightarrow} \mathbb{P}T_xX$ image $\mathcal{C}_x \subset \mathbb{P}T_xX$ is called

$[C] \longmapsto [T_xC]$ (Hwang-Mok) VMRT of X at x w.r.t. $\mathcal{K}$

Hwang-Mok's philosophy of VMRT theory:

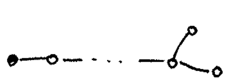
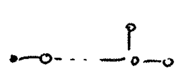
$\boxed{\text{(local) projective geometry of } \mathcal{C}_x \subset \mathbb{P}T_xX \text{ encodes (global) geometry of } X\text{, especially when } \rho_X = 1}$

Examples:

(1) $X = \mathbb{P}^n$ $\mathcal{C}_x = \mathbb{P}T_xX$ $\forall x \in X$

[Cho-Miyaoka-Shepherd-Barron] X uniruled sm. proj. with $\mathcal{C}_x = \mathbb{P}T_xX$ $x \in X$ general then $X \cong \mathbb{P}^n$

(2) $X = \mathbb{Q}^n$ $\mathcal{C}_x \simeq \mathbb{Q}^{n-2} \subset \mathbb{P}^{n-1}$

[Miyaoka, Hwang, ~~Dedieu-Hörin~~] X as above. Assume $\mathcal{C}_x \subset \mathbb{P}T_xX$ of codim. 1, then $X \cong \mathbb{Q}^n$ or projective bdle over a curve.
smooth

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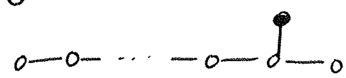
(3) $X = G/P$ P maximal associated to long root

then $\mathcal{C}_x \subset \mathbb{P}T_xX$ is again homog.

e.g.: $G/P = \mathbb{S}_n$ Spinor variety



VMRT $\simeq Gr(2,n)$

[Mok, Hong, Hwang-Hong] X sm. proj. ~~unir~~ Fano of $\rho_X = 1$

if X has the same VMRT as G/P (for some G/P. P long root)

still open for short root case → Then $X \simeq G/P$

(4) X smooth uniruled, quasi-homogeneous (e.g. toric variety etc...) horospherical

then the VMRT structure on X is isotrivial

i.e. $(\mathcal{C}_x \subset \mathbb{P}T_xX) \cong (\mathcal{C}_y \subset \mathbb{P}T_yX)$ $x, y \in X$ general.

Def: Let $Z \subset \mathbb{P}V$ be a projective subvariety

a ~~Z-on~~ isotrivial VMRT structure on X = a choice of $\mathcal{X}$ s.t. $(\mathcal{C}_x \subset \mathbb{P}T_xX) \cong (Z \subset \mathbb{P}V)$.

Question: isotrivial VMRT $\overset{?}{\Longrightarrow}$ quasi-homogeneous?

(with Z quasi-homog)

Answer: No in general

$X = \mathbb{S}_n \cap H$ VMRT $= Gr(2,n) \cap H' = Gr_\omega(2,n)$ ← rigid

$\mathrm{aut}(X) = 0$ if $n \geq 9$

($\mathbb{S}_5 \cap H$ is quasi-homog.)

$Z \subset \mathbb{P}V$

Def. A Z-isotrivial VMRT structure on X is said <u>locally flat</u> if for $x \in X$ general, $\exists$ analytic open $x \in U \subset X$ and biholomorphism $U \xrightarrow{\varphi} V$

s.t. $d\varphi : TU \to TV \simeq V \times V$ induces an isom.

$$\mathcal{C}|_U \xrightarrow[\sim]{d\varphi} V \times Z$$

<u>(Hwang-Mok's Cartan-Fubini extension thm)</u> $\rho_X = 1$ with smooth non-linear VMRT then local isom. preserving VMRT structure extends to an autom. of X.

extension to $\dim \mathcal{C}_x = 0$ case by Hwang '2016 (covered by lines)

<u>Cor</u>: $\rho_X = 1$ + locally flat VMRT $\Rightarrow$ quasi-homogeneous (equivariant compactification of $\mathbb{C}^n$).

Pb: for which $Z \subset \mathbb{P}V$, ~~a~~ Z-isotrivial VMRT ~~imp~~ must be locally flat?

[Mok] O.K. if $Z \subset \mathbb{P}V$ is the VMRT of an IHSS.

[Hwang] O.K. if $Z \subset \mathbb{P}V$ is a smooth ~~non~~ hypersurface of degree $\geq$ ~~4~~ 3

<u>Main thm (F.-Hwang '16)</u> O.K. for the following situations:

$Z \subset \mathbb{P}V$ smooth non-deg. complete intersection of type $[m_1, \cdots m_c]$ s.t.

either (i) Z is a curve of multidegree $\notin \{[3], [4], [2,2], [2,3], [2,2,2]\}$

or (ii) Z is covered by lines of multidegree $\notin \{$~~[2]~~, ~~[3]~~, $[2,2]\}$

or (iii) $Z \subset \mathbb{P}V$ is contained is a smooth hypersurface of degree $d \geq 3$ and $d = m_1 < d+2 \leq m_2 \leq \cdots \leq m_c$.

<u>Def.</u> $Z \subset \mathbb{P}V$ sm. proj. subvariety $\qquad \hat{Z} \subset V$ affine cone

$$\mathbb{H}_Z := \{\sigma : \Lambda^2 V \to V \mid \sigma(u,v) \in T_u\hat{Z} \text{ if } u \in \hat{Z} \ \ v \in T_u \hat{Z}\}$$

$$\uparrow$$

$$\mathbb{H}_V := \{\sigma : \Lambda^2 V \to V \mid \sigma(u,v) \in \mathbb{C}u + \mathbb{C}v \quad \forall u, v \in V\}$$

$$\| \wr$$

$$V^* \qquad V^* \ni \beta \longmapsto \sigma_\beta \quad \text{given by} \quad \sigma_\beta(u,v) = \beta(u)v - \beta(v)u.$$

<u>[Hwang]</u> Assume $Z \subseteq \mathbb{P}V$ smooth (~~non-deg.~~) of dim >0 s.t.

(i) $H^0(Z, T_Z \otimes \mathcal{O}(1)) = 0$

(ii) $\mathbb{H}_Z = \mathbb{H}_V$

Then any Z-isotrivial VMRT structure is locally flat.

<u>idea of proof</u> $\quad X^n$ smooth $\quad V \simeq \mathbb{C}^n \quad \mathcal{C} = Z$-isotrivial VMRT

<u>Coframe</u>: $\omega \in H^0(X, V \otimes \Omega^1_X)$ s.t. $T_x X \xrightarrow{\omega_x} V$ isom.

<u>adapted to $\mathcal{C}$</u>: ω_x sends $\mathcal{C}_x$ to $Z \subset \mathbb{P}V$.

- $\mathcal{C}$ is locally flat $\iff \exists$ a <u>conformally closed</u> adapted coframe to $\mathcal{C}$

$\forall x$, $\exists\, U \ni x \quad f$: non-vanishing s.t. $d(f\omega) = 0$

$\omega = \theta^1 e_1 + \dots + \theta^n e_n \quad \langle e_1, \dots e_n \rangle$ basis of $V \quad \theta^i$: one-form on X

$d\theta^i = \sum_{j,k} T^i_{j,k}\, \theta^j \wedge \theta^k \qquad$ structure function σ^ω $\cancel{*}$ valued in $\mathrm{Hom}(\Lambda^2 V, V)$

$$\sigma^\omega(e_j, e_k) := \sum_i T^i_{jk}\, e_i$$

ω is conformally closed $\Longleftrightarrow$ its structure function σ^ω takes values in $\mathbb{H}_V$

$H^0(Z, T_Z \otimes \mathcal{O}(1)) = 0 \Rightarrow$ uniqueness of conic connections

ω adapted to $\mathcal{C} \rightsquigarrow$ construct geodesic connections $\mathcal{F}^\omega$, ~~uniqueness conic~~

- if $\mathcal{F}^\omega$ is characteristic, then σ^ω takes value in $\mathbb{H}_Z$
- VMRT structure $\Rightarrow \exists$ characteristic connection $\Rightarrow \sigma^\omega$ takes values in $\mathbb{H}_Z$.

$$\left.\begin{array}{l} H^0(Z, T_Z\otimes\mathcal{O}(1)) = 0 \\ \text{VMRT structure} \end{array}\right\} \Rightarrow \omega \text{ adapted to } \mathcal{C}, \text{ then } \sigma^\omega \text{ takes values in } \mathbb{H}_Z$$

O.K.

To prove our main theorem, we need to check these two conditions

$Z \subset \mathbb{P}V = \mathbb{P}^N$ sm. non-deg. complete intersection of degrees $[m_1, \dots m_c]$

Prop $H^0(Z, T_Z \otimes \mathcal{O}(1)) \neq 0$ if and only if

either $\dim Z = 1$ and of multidegree $\underbrace{[2], [3], [4], [2,2], [2,3], [2,2,2]}_{N+2-\sum_{i=1}^{N-1} m_i \geq 0}$

or $\dim Z \geq 2$ with multidegree $[2], [3], [2,2]$.

idea: $\dim Z = 1$, $H^0(Z, T_Z \otimes \mathcal{O}(1)) = H^0(Z, \mathcal{O}_Z(N+2-\sum_{i=1}^{N-1} m_i)) \neq 0 \Leftrightarrow N+2-\sum_{i=1}^{N-1} m_i \geq 0$.

$\dim Z \geq 2$, $H^0(Z, T_Z \otimes \mathcal{O}(1)) \simeq H^0(Z, \Omega_Z^{n-1}(1) \otimes K_Z^*) = H^0(Z, \Omega_Z^{n-1}(N+2-\sum_{i=1}^{N-1} m_i))$

[Brückmann] if $1 \leq r \leq n-1$, then $H^0(Z, \Omega_Z^r(p)) = 0$ for $p \leq r$

(here $N+2-\sum_{i=1}^{c} m_i < n-1$, i.e. $\sum_{i=1}^{c}(m_i - 1) \geq 3$.

Computing $\mathbb{H}_Z$

$a \in Z \qquad T_a Z \simeq \mathrm{Hom}(\hat{a}, T_a\hat{Z}/\hat{a}) \qquad T_Z \otimes \mathcal{O}(-1)_a \simeq T_a\hat{Z}/\hat{a}$

$$T_Z \otimes \Omega_Z \otimes \mathcal{O}(1)_a = \mathrm{Hom}(T_Z \otimes \mathcal{O}(-1), T_Z)_a = \mathrm{Hom}(T_a\hat{Z}/\hat{a}, \mathrm{Hom}(\hat{a}, T_a\hat{Z}/\hat{a}))$$
$$\simeq \mathrm{Hom}(\hat{a} \otimes T_a\hat{Z}/\hat{a}, T_a\hat{Z}/\hat{a})$$

<u>define</u> $\zeta : \mathbb{H}_Z \longrightarrow H^0(Z, T_Z \otimes \Omega_Z(1))$

$\sigma \longmapsto \zeta(\sigma)_a(u, [v]) = [\sigma(u,v)] \qquad u \in \hat{a},\ v \in T_a\hat{Z}$

$$\mathbb{H}_Z' := \mathrm{Ker}(\zeta) = \{\sigma : \Lambda^2 V \to V \mid \sigma(\hat{a}, v) \subset \hat{a} \text{ if } a \in \hat{Z},\ v \in T_a\hat{Z}\}$$

$$\Omega_Z \otimes \mathcal{O}(1)_a = \mathrm{Hom}(T_Z \otimes \mathcal{O}(-1), \mathbb{C}) \simeq \mathrm{Hom}(T_a\hat{Z}/\hat{a}, \mathbb{C})$$

define $\mathbb{H}_Z' \xrightarrow{\eta} H^0(Z, \Omega_Z \otimes \mathcal{O}(1))$

$\sigma \longmapsto \eta_{\sigma,a} \in \mathrm{Hom}(T_a\hat{Z}/\hat{a}, \mathbb{C})$

defined by $\eta_{\sigma,a}(v)u = \sigma(u,v)$

define $\mathbb{H}_Z^0 = \mathrm{Ker}\,\eta = \{\sigma : \Lambda^2 V \to V \mid \sigma(\hat{a}, v) = 0 \ \forall a \in \hat{Z},\ v \in T_a\hat{Z}\}$

Note that $\mathbb{H}_Z^0 = 0$ if $Z \subset \mathbb{P}V$ is tangentially non-degenerate (i.e. the set of tangent lines is non-deg. in $\mathbb{P}\Lambda^2 V$)

<u>Prop</u>: Z ~~sm~~ $\subset \mathbb{P}V$ sm. complete intersection of dim > 0 then:

(i) Z is tangentially non-degenerate, hence $\mathbb{H}_Z^0 = 0$

(ii) $H^0(Z, \Omega_Z \otimes \mathcal{O}(1)) = 0$ unless Z is a curve

(iii) $\mathbb{H}_Z' \neq 0 \iff Z$ is a plane conic

Cor: (✱) Assume Z is not a plane conic, then

$$\mathbb{H}_V \subset \mathbb{H}_Z \xhookrightarrow{\ \xi\ } H^0(Z, T_Z \otimes \Omega_Z \otimes \mathcal{O}(1))$$

In particular: if $\dim H^0(Z, T_Z \otimes \Omega_Z \otimes \mathcal{O}(1)) = \dim V$

then $\mathbb{H}_Z = \mathbb{H}_V$.

hence we need to compute $\dim H^0(Z, T_Z \otimes \Omega_Z \otimes \mathcal{O}(1))$

this is very big for $Z = \mathbb{P}V = \mathbb{P}^N \rightsquigarrow N+1 + \frac{(N+2)!\,(N-1)}{2}$

$$0 \to \mathcal{O}_{\mathbb{P}V}(1) \to T_{\mathbb{P}V} \otimes \Omega_{\mathbb{P}V} \otimes \mathcal{O}(1) \to ad(T_{\mathbb{P}V})(1) = \Omega^{T_{N-1}}_{\mathbb{P}^N}(N+2) \to 0$$

[Brückmann] $\dim H^0(Z, T_Z \otimes \Omega_Z \otimes \mathcal{O}(1)) = \dim V$ if Z is a hypersurface of degree ≥ 3.

Case(iii) $Y \subset \mathbb{P}V$ sm hypersurface of degree $d \geq 3$

$Z \subset Y$ complete intersection of type $[m_1, \cdots m_c]$

if $m_j \geq d$ then $H^0(Z, T_Y \otimes \Omega_Y(1)|_Z) \cong H^0(Y, T_Y \otimes \Omega_Y(1)) = H^0(Y, \mathcal{O}(1))$

if $m_j \geq d+2$ then $H^0(Z, T_{Y|Z}(1-m_j)) = H^1(Z, T_{Y|Z}(1-m_j)) = 0$

and $H^0(Z, T_Y \otimes \Omega_Y(1)|_Z) \simeq H^0(Z, T_{Y|Z} \otimes \Omega_Z(1))$

$$0 \to T_Z \to T_{Y|Z} \to N_{Z|Y} \to 0$$

get $0 \to T_Z \otimes \Omega_Z(1) \to T_{Y|Z} \otimes \Omega_Z(1) \to N_{Z|Y} \otimes \Omega_Z(1) \to 0$

$$H^0(T_Z \otimes \Omega_Z(1)) \hookrightarrow H^0(T_{Y|Z} \otimes \Omega_Z(1)) \leftarrow \dim = \dim V$$

o.k.

<u>Case (ii)</u> : $Z \subsetneq \mathbb{P}V$ covered by lines, not a hyperquadric

then $H^0(Z, T_Z \otimes \Omega_Z(1)) \simeq H^0(Z, \mathcal{O}(1))$

<u>idea</u> :
$$0 \to \mathcal{O}_Z(1) \to \underset{\underset{\textstyle End(T_Z)(1)}{\|}}{T_Z \otimes \Omega_Z(1)} \xrightarrow{\gamma} ad(T_Z)(1) \to 0$$

Take $A \in H^0(Z, T_Z \otimes \Omega_Z(1))$

$A_x \in End(T_x Z)$

$C \subset Z$ a line $\qquad T_Z|_C \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$

$a = [T_x C] \in \mathcal{C}_x \qquad T_a \hat{\mathcal{C}}_x \simeq (\mathcal{O}(2) \oplus \mathcal{O}(1)^p)_x$

A restricts to C gives: $\underset{\underset{\textstyle \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q}{\|}}{T_Z|_C} \xrightarrow{A|_C} T_Z|_C \otimes \mathcal{O}(1) = \mathcal{O}(3) \oplus \mathcal{O}(2)^p \oplus \mathcal{O}(1)^q$

$a \in \mathcal{O}(2) \xrightarrow{A_x} \mathcal{O}(3) \oplus \mathcal{O}(2)^p \in T_a \hat{\mathcal{C}}_x$

<u>hence</u> $A_x(a) \in T_a(\hat{\mathcal{C}}_x) \quad \forall x, a$

$\Rightarrow A_x \in aut(\hat{\mathcal{C}}_x)$

Z C.I. $\Rightarrow$ so is $\mathcal{C}_x \Rightarrow aut(\hat{\mathcal{C}}_x) = \mathbb{C}$ unless Z is $\mathbb{P}V$ or hyperquadric

$\Rightarrow A$ has no traceless part.

$\Rightarrow \gamma(A) = 0 \Rightarrow A \in H^0(\mathcal{O}_Z(1))$

O.K.