"Integrable deformations of analytic fibrations with singularities"

BRUNO SCÁRDUA (UFRJ)

Joint work with

Dominique CERVEAU (Rennes/FR)

Jan. 09/2017

NUS Conf.

Some History:

H. Poincaré: Search for conditions to assure the "integrability" of a differential equation. More precisely:

A differential equation of the form

$$(*) \quad \frac{dy}{dx} = \frac{A(x,y)}{B(x,y)}$$

, A, B holomorphic or polynomial complex functions.

<u>local framework</u>: A, B holomorphic in a neighborhood of the origin $(0,0) \in \mathbb{C}^2$ which is a singular point: $A(0,0) = B(0,0) = 0$

Then search for conditions that assure the existence of a holomorphic or meromorphic <u>1st integral</u> $f : U \subset \mathbb{C}^2 \longrightarrow \overline{\mathbb{C}}$. That is, a function constant along the solutions of $(*)$

To each equation $(*)$ we associate a holomorphic vector field

$$X(x,y) = B(x,y)\frac{\partial}{\partial x} + A(x,y)\frac{\partial}{\partial y}$$

and/or a holomorphic one-form

$$\omega(x,y) = A(x,y)\,dx - B(x,y)\,dy .$$

We may think of $(*)$ as defining a holomorphic foliation $\mathcal{F}_\omega$ of (co)dimension one with a singularity at $(0,0) = \operatorname{sing}(\mathcal{F}_\omega)$.

In this sense we have:

Theorem (Mattei-Moussu, 1983):
A germ of foliation $\mathcal{F}_\omega$ at $0 \in \mathbb{C}^2$ admits a holomorphic first integral if and only if:

(i) All leaves are closed outside of the origin $0 \in \mathbb{C}^2$

(ii) Only a finite number of the leaves accumulate at the origin.

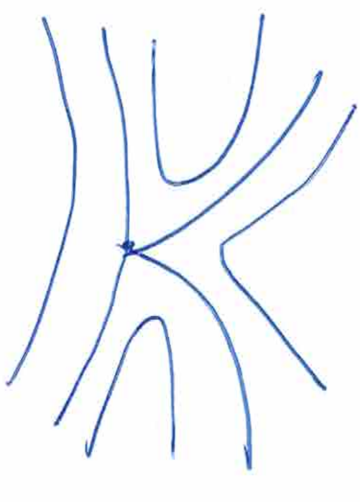
In other words (taking into account Remmert-Stein Extension Theorem):

$\mathcal{F}_\omega$ admits a holomorphic 1^{st} integral

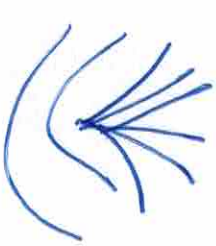
$$\Updownarrow$$

(i) Every leaf is (contained in an) analytic (curve)

(ii) Only a finite number of ("leaves") invariant analytic curves pass through the origin (non dicritical singularity).



(non-dicritical)



(dicritical)

In this sense we have the following version for the

Global framework:

Theorem (Darboux): Given a polynomial 1-form $\omega = P\,dx + Q\,dy$ on $\mathbb{C}^2$ then $\mathcal{F}_\omega$ admits a rational first integral if, and only if, $\mathcal{F}_\omega$ exhibits infinitely many algebraic invariant curves.

Remark: (i) $\mathcal{F}_\omega$ extends to a foliation by Riemann surfaces on $\mathbb{C}P(2)$.

(ii) By Remmert-Stein extension theorem and Chow's theorem a leaf ~~of~~ of $\mathcal{F}_\omega$ on $\mathbb{C}P(2)$ is (contained in an) algebraic (invariant curve) iff it is closed off the singular set of $\mathcal{F}_\omega$ on $\mathbb{C}P(2)$.

Mattei-Moussu and Darboux theorem above are indeed for codimension one foliations.

We consider a different problem.

Starting with a holomorphic function germ $f: \mathbb{C}^n, 0 \longrightarrow \mathbb{C}, 0$, $n \geq 3$. We consider an analytic deformation of the foliation $\mathcal{F}_0 : \omega_0 = df = 0$ by level hypersurfaces of f.

The deformation being written as

$\omega_t = \omega_0 + t\omega_1 + t^2\omega_2 + \cdots$ where $t \in \mathbb{C}, 0$,

the ω_j are holomorphic in some small neighborhood U of $0 \in \mathbb{C}^n$, $n \geq 3$ $\forall j$ and

$\omega_0 = df_0$

ω_t satisfies the integrability condition, i.e., $\ker(\omega_t)$ defines a foliation.

Question: Under which conditions can we assure the existence of a holomorphic first integral for ω_t?

Theorem A: Assume that the germ $f \in \mathcal{O}_n$ is REDUCED, $X_f : (f=0)$ is irreducible and has only normal crossings outside of a codimension ≥ 3 subset.
Let $\{\omega_t\}_{t \in \mathbb{C},0}$ be an integrable analytic deformation of $\omega_0 = df$ at $0 \in \mathbb{C}^m$, $m \geq 3$.

Then for any $t \in (\mathbb{C},0)$ close to 0, the one-form ω_t admits a holomorphic 1st integral.

Indeed, there is a germ of a holom: function $F : (\mathbb{C}^n \times \mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$

$(x,t) \longmapsto F(x,t)$ such that:

1. $F(x,0) = f(x)$
2. $F_t : x \longmapsto F(x,t)$ is a first integral for ω_t.

As corollaries we obtain:

~~COROLLARY A~~:

<u>Theorem B</u>: Let $f \in \mathcal{O}_n$, $n \geq 3$ be a strictly <u>quasi-homogeneous</u> reduced function. Assume that X_f is irreducible with normal singularities off a cod ≥ 3 subset. Then any integrable one-form germ $\omega = df + f\eta$ (η hol 1-form germ at $0 \in \mathbb{C}^n$), admits a holomorphic 1^{st} integral,

Rk : f is strictly quasi-homogeneous
if $\exists\ d, d_1, \ldots, d_n \geq 0$ s.t.
$f(t^{d_1}x_1, \ldots, t^{d_n}x_n) = t^d f(x_1, \ldots, x_n)$ and
~~some~~ $d_j > 0, \forall j$ and $d > 0$.

Theorem G : (Isolated singularity case)
Let $f \in \mathcal{O}_n$ be reduced and irreducible with an ISOLATED SINGULARITY at $0 \in \mathbb{C}^n$, $n \geq 3$.
Let ω be a holomorphic 1-form germ at $\underline{0}$, integrable and having X_f as the only invariant hypersurface.
Then: ω admits a hol 1st integral

$$\Updownarrow$$

$$\nu(\omega) = \nu(df)$$ (algebraic multiplicities at the origin)

NORMAL CROSSINGS Hypersurface germs

Theorem (Lê-Saito): $f \in \mathcal{O}_n$ reduced. Let $n \geq 3$. Assume that outside of an analytic subset $(Y,0) \subset (X_f,0)$ of dimension at most $n-3$, the only singularities of $(X_f,0)$ are normal crossings. Then the local fundamental group of the complement of $(X_f,0)$ in $(\mathbb{C}^n,0)$ is abelian. The Milnor fiber of f has a fundamental group which is free abelian of rank the number of analytic components of X_f at 0, minus one. Finally, if X_f is irreducible, then the fiber $f^{-1}(c)$, $c \neq 0$ is SIMPLY-CONNECTED.

The Integrability condition:

$$\omega_t = \omega_0 + t\omega_1 + \ldots + t^k\omega_k + \ldots$$

- integrable analytic deformation of $\omega_0 = df_0$

where

$f_0 \in \mathcal{O}(n)$ is reduced, X_{f_0} is irreducible with normal crossings singularities off a cod. ≥ 3 subset.

The integrability condition $\omega_t \wedge d\omega_t = 0$ gives a set of equations:

$$\omega_0 \wedge d\omega_0 = 0$$
$$\omega_0 \wedge d\omega_1 + \omega_1 \wedge d\omega_0 = 0$$
$$\omega_2 \wedge d\omega_0 + \omega_1 \wedge d\omega_1 + \omega_0 \wedge d\omega_2 = 0$$
$$\vdots$$

Since $\omega_0 = df_0$ we have

$$df_0 \wedge d\omega_1 = 0. \quad \text{and}$$

$$\omega_1 \wedge d\omega_1 + df_0 \wedge d\omega_2 = 0.$$

Then we make use of the:

Lemma (Relative Cohomology Lemma):

Under the above hypotheses we have:

$$d\omega_1 \wedge df_0 = 0 \Longrightarrow \omega_1 = df_1 + a_1\, df_0$$

for some holomorphic functions f_1, a_1 in U.

Then we rewrite

$$\begin{aligned}
\omega_t &= \omega_0 + t\omega_1 + t^2\omega_2 + \cdots \\
&= df_0 + t(df_1 + a_1\, df_0) + t^2\omega_2 + \cdots \\
&= (1 + ta_1)\, df_0 + t\, df_1 + t^2\omega_2 + \cdots
\end{aligned}$$

Therefore

$$\frac{1}{1+ta_1}\omega_t = df_0 + \frac{t}{1+ta_1} df_1 + \frac{t^2}{1+ta_1}\omega_2 + \cdots$$

$$= d(f_0 + tf_1) + t^2(\cdots)$$

Put $\tilde{\omega}_t = \frac{1}{1+ta_1}\omega_t$. Recall that:

- $1+ta_1$ is a unit
- $\tilde{\omega}_t$ is also integrable

$$\tilde{\omega}_t = df_0 + t\,df_1 + t^2\tilde{\omega}_2 + \cdots$$

$$d\tilde{\omega}_t = t^2 d\tilde{\omega}_2 + \cdots$$

From $\tilde{\omega}_t \wedge d\tilde{\omega}_t = 0$ we obtain

$df_0 \wedge d\tilde{\omega}_2 = 0$ and as above

$\tilde{\omega}_2 = df_2 + a_2\, df_0$ for some holom. $a_2, f_2 : U \to \mathbb{C}$

Then

$$\tilde{\omega}_t = df_0 + t\,df_1 + t^2(df_2 + a_2\,df_0) + t^3\tilde{\omega}_3 + \cdots$$

$$= (1+t^2a_2)\,df_0 + t\,df_1 + t^2 df_2 + t^3\tilde{\omega}_3 + \cdots$$

$$\tilde{\tilde{w}}_t := \frac{1}{1+t^2 a_2}\tilde{w}_{\cancel{t}} = df_0 + t\,df_1 + t^2 df_2 + t^3 \tilde{\tilde{w}}_3 + \cdots$$

Inductively we obtain a formal unit $\hat{G}$ such that

$$\frac{1}{\hat{G}} w_t = df_0 + \sum_{j=1}^{\infty} t^j df_j \quad \text{for some}$$

holomorphic functions $f_j : U \to \mathbb{C}$, $j \geq 1$.

Thus we can write

$$w_t = \hat{G}(x,t) . d_x \hat{F}(x,t) \quad \text{in the obvious sense}$$

where $\hat{F}(x,t) = f_0 + \sum_{j \geq 1}^{\infty} t^j f_j$ is a formal function. Put $\Omega(x,t) = w_t(x)$. ~~Since~~

$\cancel{\Omega = \hat{G}\, d\hat{F}}$

Claim: { ~~So~~ We have

$$\{\Omega, dt\} = \{d\hat{F}, dt\}$$ at the level of formal modules.

Indeed, there is a formal series $\hat{h}(x,t)$ such that $\omega_t + \hat{h}\,dt = \hat{G}\,d_{(x,t)}\hat{F}$.

So $\Omega = \hat{G}\,d\hat{F} - \hat{h}\,dt$.

Also, the pair $\{\Omega, dt\}$ is integrable:

$$d_{(x,t)}\Omega = d_x\omega_t + \frac{\partial\omega_t}{\partial t}dt \quad \text{so that}$$

$$\Omega\wedge d\Omega\wedge dt = \omega_t\wedge d_x\omega_t\wedge dt + \omega_t\wedge\frac{\partial\omega_t}{\partial t}dt\wedge dt \equiv 0$$

because we have $\omega_t\wedge d_x\omega_t \equiv 0$.

<u>Malgrange's Theorem</u> (<u>Singular Frobenius</u>)

Let be given p germs of holom. one-forms $\omega_1, \ldots, \omega_p \in \Omega^1(\mathbb{C}^n, 0)$, $1 \le p \le n$. Denote by S the germ of analytic set given by $\omega_1\wedge\cdots\wedge\omega_p = 0$.

Assume that $\operatorname{cod} S \geq 2$ and $\{\omega_1, \ldots, \omega_p\}$ is <u>formally integrable</u> ($\exists\ \hat{f}_i, \hat{g}_{ij} \in \hat{\mathcal{O}}_n$ s.t. $\omega_i = \sum_j \hat{g}_{ij}\,d\hat{f}_j$ and $\det(\hat{g}_{ij}(0)) \neq 0$)

Then $\{\omega_1, \dots, \omega_p\}$ is <u>fully integrable</u>

$(\exists f_i, g_{ij} \in \mathcal{O}_n$ (convergent)$)$ s.t.

$$\omega_i = \sum_j g_{ij}\, df_j \qquad (\det g_{ij}(0) \neq 0).$$

Applying to $\{\Omega, dt\} = \{d\hat{F}, dt\}$ we conclude that $\{\Omega, dt\}$ is fully integrable:

indeed, because $\Omega \wedge dt = d_x f_0 \wedge dt + \sum_{j \geq 1} t^j \omega_j \wedge dt$

and since $\operatorname{cod} \operatorname{sing}(df_0) \geq 2$ we have $\operatorname{cod} \operatorname{sing}(\Omega \wedge dt) \geq 2$.

Thus $\{\Omega, dt\} = \{dF, dt\}$ for some holom. funct. $F(x,t) \in \mathcal{O}_{n+1}$ s.t.

$$\Omega = a \cdot d_x F(\cdot, t) + b\, dt$$

a, b hol., $\underline{a}$ unit.

$$\therefore \quad \omega_t = a\, d_x F(x,t).$$

RELATIVE COHOMOLOGY LEMMA:

$f \in \mathcal{O}_n, n \geq 3.$

LEMMA 1: Assume that $\operatorname{codsing}(df) \geq 2$. Then the following are equivalent:

(i) $d\omega \wedge df \equiv 0$

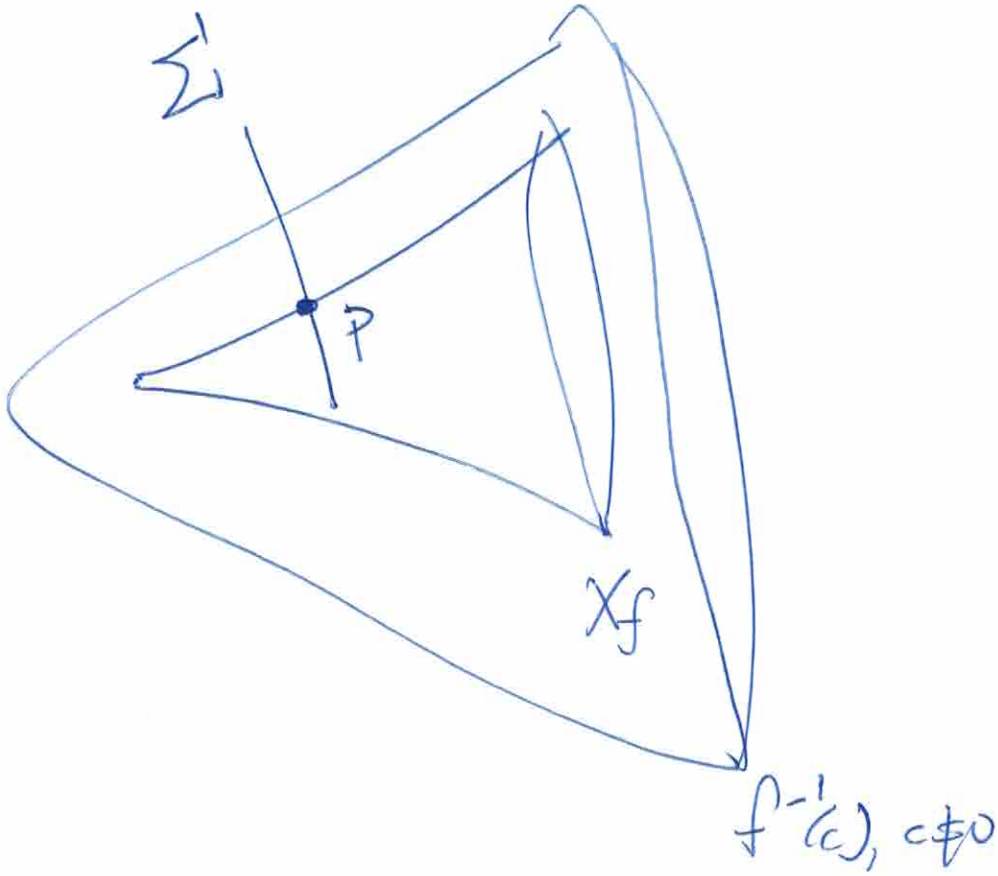
(ii) ω is closed on each fiber of f.

Proposition: Assume that $\operatorname{codsing}(f) \geq 2$. For $\omega \in \Omega^1(\mathbb{C}^n, 0)$ the following are equivalent:

(i) $d\omega \wedge df = 0$ and $\oint_\gamma \omega \equiv 0$, $\forall$ cycle γ contained in a non-singular fiber $f^{-1}(c)$, $c \neq 0$ of f.

(ii) $\omega = a\,df + dh$ for some $a, h \in \mathcal{O}_n$.

Uses Riemann Extension theorem.



$p \in \underbrace{X_f \setminus \mathrm{sing}(X_f)}_{\text{connected}}$

$\bigcup f^{-1}(f(z))$ = neighbd of the origin minus X_f.

$z \in \Sigma \setminus \{p\}$

Define h on Σ by $h(p)=0$, $h(z)=f(z)$, $\forall z \in \Sigma$.

Extend h to each fiber $f^{-1}(f(z))$, $z \neq p$ by integration

$$h(w) = \overset{= f(z)}{h(z)} + \int_z^w \omega\Big|_{f^{-1}(f(z))}$$

References:

[1] M. Berthier, D. Cerveau: Ann. Sc. Ec. Norm. Sup. (1993), Vol. 26, 3, 403-424.

[2] Lê-Saito, "The local π_1 of the complement of hypersurfaces with normal crossings in codimension one is abelian", Ark. Math. 22 (1984), nº 1, 1-24.

[3] B. Malgrange: "Frobenius avec singularites 2. Le cas general". Inv. Math. (1977), 39, 67-90.

Bruno Scárdua
Inst. of Math. CP 68530
Federal University of Rio de Janeiro
21945-970 Rio de Janeiro-RJ
BRAZIL.

scardua@im.ufrj.br