Dynamical Aspects of Hamiltonian System

CHENG, Chong-Qing

Department of Mathematics, Nanjing University

Workshop on Higher Dimensional Algebraic Geometry,
Holomorphic Dynamics and their Interactions
January 18, 2017 - Singapore

Hamilton-Jacobi Equation

Hamilton-Jacobi equation

$$H(x, \partial_x U) = E, \qquad x \in \mathbb{R}^n.$$

Its characteristic equation (Hamiltonian equation)

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \qquad i = 1, 2, \cdots n.$$

If $U \in C^1(\mathbb{R}^n, \mathbb{R})$ solves the Hamilton-Jacobi equation,

$$\mathbb{L}_U = \Big\{(x, \partial_x U(x)) : x \in \mathbb{R}^n\Big\}$$

is called the Lagrange graph, it admits a foliation of orbits of the Hamiltonian flow Φ_H^t.

On $T^*\mathbb{R}^n$, there exists a natural symplectic form $\omega = d\alpha$, where

$$\alpha = \sum y_i dx_i$$

Lagrange sub-manifold $\mathbb{L}$ is an n-dimensional manifold such that $\omega|_{\mathbb{L}} = 0$, or, equivalently

$$\oint_\gamma \alpha = 0, \qquad \forall \text{ closed path } \gamma \subset \mathbb{L} \text{ with } [\gamma] = 0.$$

Let $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ be the standard projection, $\pi(x, y) = x$. $\mathbb{L}$ is called horizontal if π^{-1}: $\pi\mathbb{L} \to \mathbb{L}$ is injective.

$\mathbb{L}$ is a horizontal Lagrange sub-manifold if $\exists$ a smooth function F such that

$$\mathbb{L} = \text{Graph}(dF).$$

Assume the positive definite condition:

$$\frac{\partial^2 H}{\partial y_i \partial y_j} \text{ is positive definite.}$$

Convex Hamiltonian is uniquely related to a convex Lagrangian

$$L(x, \dot{x}) = \max_y \langle \dot{x}, y \rangle - H(x, y)$$

The Hamiltonian equation is equivalent the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0.$$

The Legendre transformation $\mathscr{L}$: $(x, y) \to (x, \dot{x} = \partial_y H(x, y))$.

Variational Principle

The Lagrange equation corresponds to a critical point of the action functional

$$\gamma \to \int L(\gamma(t), \dot\gamma(t))dt.$$

Each orbit $(x(t), y(t))$ on a horizontal Lagrange sub-manifold is minimal: for any $t < t'$ one has

$$\int_t^{t'} L(x(t), \dot x(t)dt = \inf_{\substack{\xi(t)=x(t) \\ \xi(t')=x(t')}} \int_t^{t'} L(\xi(t), \dot\xi(t)dt.$$

The Weierstrass's principle:

$$L \to L' = L - \left\langle \frac{\partial U}{\partial x}, \dot x \right\rangle.$$

$\forall\ x \in \mathbb{R}^n$, L' reaches its minimal at $\mathscr{L}(\pi^{-1}x)$.

Closed Configuration Manifold

An interesting case: H is periodic in $x \Rightarrow x \in \mathbb{T}^n$. If U solves the Hamilton-Jacobi equation

$$U(x) = u(x) + \langle c, x \rangle$$

where u is periodic in x. The Weierstrass's principle:

$$L \to L' = L - \Big\langle \frac{\partial u}{\partial x} + c, \dot{x} \Big\rangle.$$

The Lagrange equation for L' is the same as for L.

Does there exist classical solution of the Hamilton-Jacobi equation?

$$H(x, \partial_x U) = E, \qquad x \in \mathbb{R}^n.$$

The problem was considered by G.D. Birkhoff. A geodesic flow on closed Riemann surface M with negative curvature.

By Gauss-Bonnet formula, if $g(M) \geq 2 \Rightarrow$ it is possible that M is endowed with a curvature negative everywhere.

Birkhoff considered a closed surface with $g(M) = 2$, illustrated the proof of "topological instability". Indeed, with constant negative curvature, the system is uniformly hyperbolic, Anosov system.

Anosov system is totally non-integrable $\Rightarrow$ there does not exist Lagrangian graph.

However, torus can not be endowed with curvature negative everywhere.

An Example of Non-existence for $\mathbb{T}^n$

It is non-trivial to consider the problem if $E \geq \min \alpha(c)$, Mañé critical value.

$$\alpha(c) = -\min_{\mu} \int (L - \eta_c) d\mu.$$

$\alpha(c)$ is the homogenized Hamiltonian for H. A classical system (mechanical system)

$$H(x, y) = \frac{1}{2}\langle Ay, y\rangle + V(x)$$

where A is positive definite and $\max V = 0 \Rightarrow \min \alpha = 0$

Theorem (C-2015, Zhou-2016) For C^r-generic potential V $(r = 2, 3, \cdots, \infty, \omega)$, $\exists\ d > 0$ such that $\forall\ E \in [0, d]$ the equation

$$\frac{1}{2}\langle A\partial_x U, \partial_x U\rangle + V(x) = E$$

does not have classical solution.

However, on high energy level set, there are many invariant Lagrange Graphs: KAM tori if $H \in C^r$ with $r > 2n$. By rescaling

$$y \to \sqrt{E}y, \qquad H \to \frac{1}{E}$$

One obtains from the classical system a new one which is a small perturbation of integrable system

$$H = \frac{1}{2}\langle Ay, y\rangle + \frac{1}{E}V(x).$$

Given a Diophantine frequancy $\omega \in \mathbb{R}^n$, there exists smooth solution $U(x) = u(x) + \langle A^{-1}\omega, x\rangle$ with small u.

KAM theory (Kolmogorov, Arnold and Moser)

For nearly integrable Hamiltonian

$$H(x,y) = h(y) + \epsilon P(x,y), \qquad (x,y) \in \mathbb{T}^n \times \mathbb{R}^n,$$

1. $h, P \in C^r$ $(r = 2n + \delta, \cdots, \infty, \omega)$;
2. the Hessian of h is non-degenerate;
3. $\omega^* = \partial^{-1} h(y^*)$ is Diophantine, namely,

$$|\langle \omega^*, k \rangle| \geq D|k|^{-\mu}, \qquad k \in \mathbb{Z}^n \backslash \{0\}$$

there exists smooth solution $U(x)$ and small $u(x)$ such that the invariant torus

$$\partial_x U(x) = y^* + \partial_x u(x)$$

admits a foliation of quasi-periodic orbits with the frequency ω^*.

Dynamical stability

For $n = 2$, one obtains from KAM theory the dynamical stability: 2-d invariant tori separate 3-d energy level set.

$n = 2 \Rightarrow$ area-preserving twist map: the Lyapunov stability of elliptical fixed point.

Application: the stability of Greek and Trojan asteroids.

For $n \geq 3$, n-torus does not separate $(2n - 1)$-d energy level set, the complementary part of KAM tori is path-connected, $\Rightarrow$ whether it is dynamically connected? Arnold diffusion (1963).

For volume preserving map $(\phi_1, \phi_2, r) \to (\phi_1', \phi_2', r')$

$$
\begin{aligned}
\phi_1' =& \phi_1 + h_1(r) + \epsilon \Phi_1(\phi_1, \phi_2, r), \\
\phi_2' =& \phi_2 + h_2(r) + \epsilon \Phi_2(\phi_1, \phi_2, r), \\
r' =& r + \epsilon R(\phi_1, \phi_2, r)
\end{aligned}
$$

there are a lot of 2-dimensional tori (C-Sun 1990). It can be generalized to $n+1$ for arbitrary n (Herman, Xia).

A consequence of the theorem is the dynamical stability: quasi-ergodic hypothesis does not hold for volume-preserving maps.

Various generalizations of KAM-2

For nearly integrable Hamiltonian

1. for twist map ($n = 2$), Moser required $r = 333$, for arbitrary n, Arnold assumed analytical condition. It is finally reduced to $r > 2n$.
2. Kolmogorov's condition: $\partial^2_{yy} h \neq 0 \Rightarrow \partial h$ maps an n-ball to a set containing a twist curve (weakest ondition, C-1994).
3. non-degenerate condition is not satisfied in n-body problem. It was solved by Chierchia-Pinzari (2011).

Destruction of KAM tori

Smoothness seems to be the crucial condition for the existence of KAM torus.

1. for $n = 2$: C^r with $r < 4$ no (Herman 1983, local analysis)
2. for arbitrary $n > 2$: $r < 2n$ no (C.-Wang 2013 Lagrange graph, variational approach)

It is interesting that to obtain C^1-solution of the Hamilton-Jacobi equation we need to assume $H \in C^r$ with $r > n$

Lagrange graph is thought (KAM tori) as obstacle of dynamical instability. However

for $n \geq 3$, Arnold said "It can be supposed that topological instability is a typical case: certain trajectories beginning in the gaps between invariant tori can go a long way, since all the gaps merge into a connected set extending to infinity (see Fig. 18)"

Fig. 18.

In 1964, Arnold constructed an example

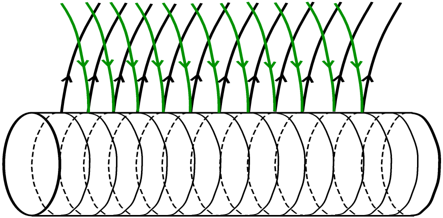
$$H = \frac{1}{2}(y_1^2 + y_2^2) + \epsilon(1 - \cos x_1)(1 + \mu(\cos x_2 + \cos t)).$$

For any $A < B$, there exist orbits of the Hamiltonian flow which connect the region $\{y_2 < A\}$ to the region $\{y_2 > B\}$. However, this example is not typical (generic).

Arnold's example is so-called *a priori* unstable system, it has a normally hyperbolic cylinder invariant for the time-2π-map.

$$\Gamma = \{(x_1, x_2, y_1, y_2) \in \mathbb{T}^2 \times \mathbb{R}^2 : (x_1, y_1) = 0\}$$

The perturbation is carefully constructed so that dynamics on the cylinder is still integrable, i.e. the cylinder admits a foliation of circles invariant for the time-2π-map



A priori unstable system and gap problem

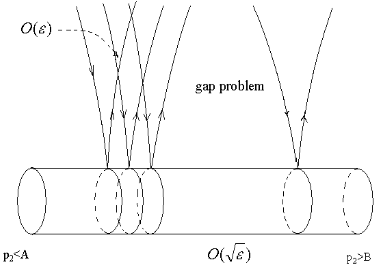
A general form of Arnold's example is *a priori* unstable systems (perturbation of rotator+pendulum)

$$H = h_1(p) + h_2(x, y) + P(p, q, x, y, t) \qquad (x, y) \in \mathbb{T}^k \times \mathbb{R}^k$$

and $(p, q) \in \mathbb{R}^\ell \times \mathbb{T}^\ell$, $\ell = 1$ for time-periodic and $\ell = 2$ for autonomous. $(x, y) = 0$ is a hyperbolic fixed point for the Hamiltonian flow $\Phi^t_{h_2}$, it corresponds to a normally hyperbolic cylinder invariant for the flow $\Phi^t_{h_1+h_2}$.

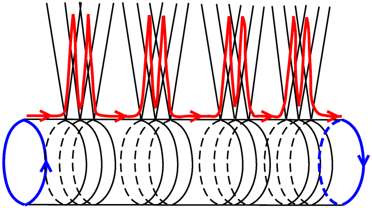
The cylinder survives small perturbation P, but the dynamics on the cylinder is no longer integrable for the time-2π-map, it is a twist map. Strong resonance induces Birkhoff instability region.

The size of the Birkhoff instability region is of the order $O(\sqrt{\epsilon})$, the splitting of the stable and unstable manifold is at most of order $O(\epsilon)$.



It puzzled people for almost 40 years.

Theorem (C.-Yan 04) It is C^r-generic with $(r \geq 4)$ that there exist diffusion orbits along the cylinder.



There are also the works of D. Treschev (04), de la Llave et al (06), Bernard (09).

Arnold diffusion for nearly integrable system

J. Mather announced in 2003 for $n = 3$

C. arXiv 1207.4016v2 (2012)

Let $H(x, y) = h(y) + \epsilon P(x, y)$ with $(x, y) \in \mathbb{T}^3 \times \mathbb{R}^3$. Assume $H \in C^r$ $(r \geq 6)$ and $E > \min h$.

Then, for any finite small balls $B_\delta(y_k) \subset h^{-1}(E)$ $(k = 1, \cdots, m)$, it is C^r-generic that Φ_H^t has orbits such that its action-component passing through these balls.

C.-Xue arXiv1503.04153 (2015)

The above result still holds for arbitrary n for C^r with $r \geq 2n$.

There are also the works of Kaloshin & Zhang and of Marco, only for $n = 3$.

Is KAM torus dynamically stable?

No! There exists a class of nearly integrable systems in which a prescribed KAM torus is dynamically unstable, there exist orbits which do not stay on the torus, but take the torus as their ω-limit set (α-limit set). (C-Zhang 2014).

Speed of diffusion

In energy level set of nearly integrable Hamiltonian system, KAM invariant tori occupy a non-where dense set with nearly full Lebesgue measure.

The complementary part is open-dense and path connected. The existence of Arnold diffusion implies it is also dynamically connected.

However, the speed of diffusion is extremely slow, as one has Nekhoroshev estimate

$$|y(t) - y(t_0)| < \epsilon, \qquad \forall\ |t - t_0| < Ce^{-\frac{1}{\epsilon^\sigma}}.$$

The stability of the solar system. It is nearly integrable with $\epsilon = 5 \times 10^{-4}$.

Quasi-ergodic hypothesis

Almost during the same time of Poincaré, Ludwig Boltzmann considered a closely related problem from different point of view: statistical physics

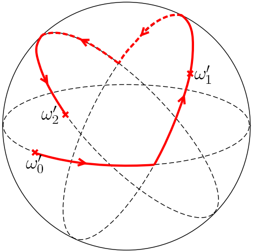
1. **ergodic hypothesis**: average over time = average over phase space

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k} f(T^i x) = \int f d\mu, \qquad \forall\ f \in L^1.$$

(quite often, Birkhoff ergodic theorem is not applicable in natural problems)

2. **quasi-ergodic hypothesis**: dense orbit in energy level set or the whole phase space (P. Ehrenfest, G. D. Birkhoff, M. Herman)

Diffusion emerges in resonant layers. For $H = h(y) + \epsilon P(x, y)$,



Along the path, there are points of strong double resonance. Outside of certain neighborhood of double resonance, it turns out to be a priori unstable system.

Dynamics around double resonance

The reduced system is a time-periodic perturbation of a classical systems (A is positive definite, $(p, q) \in T^*\mathbb{T}^2$)

$$G_\epsilon(p, q, t) = \frac{1}{2}\langle Ap, p\rangle + V(q) + \sqrt{\epsilon}R_\epsilon\Big(p, q, \frac{t}{\sqrt{\epsilon}}\Big).$$

Rescaling $y \approx \sqrt{\epsilon}p$, $\sqrt{\epsilon}$-neighborhood of double resonant point $\Rightarrow$

1. for truncated system $\frac{1}{2}\langle Ap, p\rangle + V(q)$, does there exists NHIC consists of periodic orbits with class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, the period ranges $(0, \infty)$
2. under time-periodic perturbation, how close the cylinders get to the fixed point?
3. how to dynamically join different cylinders around the fixed point.

An Aubry set for a class $c \in H^1(\mathbb{T}^n, \mathbb{R})$ is dynamically connected to another one for c' with $|c - c'| \ll 1$, if

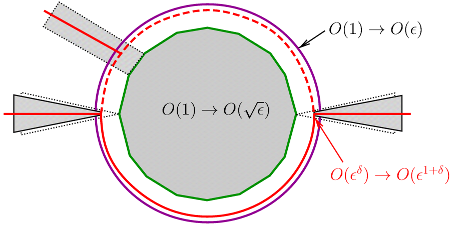
$\exists$ a section $\mathcal{A}_0(c)$ of the Aubry set which is transversal to each curve in the set such that

$$\oint_\gamma \mu_{c'-c} = 0, \qquad \forall \text{ closed curve } \gamma \subset \mathcal{A}_0(c).$$

where the 1-form is closed such that $[\mu_{c'-c}] = c' - c$.

The way to cross double resonance

The fixed point for Φ_ϵ (the periodic orbit for the flow) is surrounded by "stochastic layer" where one is able to establish cohomology equivalence: the circles of equivalence make up an annulus surrounding $\alpha^{-1}(\min \alpha)$.



As single resonant channel extends into the annulus of equivalence, any two channels are connected by circles of equivalence. (C 2015)

Dynamical Implication of weak KAM solution

Although the Hamilton-Jacobi equation is not integrable in general, for Hamiltonian convex in y, there exist "good" weak solutions (viscosity solution, or weak KAM solution) which are Lipschitz, although classical solution does not exist.

Given a cohomology class $c \in H^1(\mathbb{T}^n, \mathbb{R})$, there are two types of weak KAM solutions

1. backward weak KAM U_c^-: at each differentiable point x, $(x, \partial_x U_c^-(x))$ determines an orbit approaching the Mather set for the cohomology class c as $t \to -\infty$;
2. forward weak KAM U_c^+: at each differentiable point x, $(x, \partial_x U_c^+(x))$ determines an orbit approaching the Mather set for the cohomology class c as $t \to \infty$.

Given $c \in H^1(\mathbb{T}^n, \mathbb{R})$, we consider weak KAM solutions of the Hamilton-Jacobi equation

$$H(x, \partial_x u^\pm + c) = \alpha(c).$$

We are interested in elementary weak KAM solution.

Whether it can be re-parameterized by some σ such that

$$|u^\pm_{c(\sigma)} - u^\pm_{c(\sigma')} \leq C|\sigma - \sigma|^\nu$$

where $\nu > 0$.

It is true for $n = 2$ and $\nu = \frac{1}{3}$ (C-Xue 2016).

Simultaneous hyperbolicity

Let $G_0 = \frac{1}{2}\langle Ap, p\rangle + V(q)$. For $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ and $E > 0$, $\exists$ minimal periodic orbit on $G_0^{-1}(E)$, which corresponds to the minimal point of $F(\cdot, E) : \mathbb{T} \to \mathbb{R}$

$$F(x, E) = \inf_{\gamma(0)=\gamma(2\pi)=x} \int_0^{2\pi} \bar{L}(d\gamma(\tau), \tau, E)d\tau.$$

C-Zhou

1. it may not be smooth (conjugate point), but it is smooth around its minimal point;
2. for C^r-generic potential V with $r \geq 5$, it holds simultaneously that the minimum of $F(\cdot, E)$ is non-degenerate $\forall$ $E \in (0, \infty)$;
3. on bounded $[E, E']$, except for finitely many E_i, the minimal point is unique, for $E = E_i$ there are two minimal points.

Simultaneously transversality of intersections

with the simultaneous hyperbolicity, one obtains a normally hyperbolic invariant cylinders for the flow.

Under time-periodic perturbation, the cylinder persists for the time-periodic map $\Phi_H^t|_{t=2\pi}$.

1. On a normally hyperbolic invariant cylinder invariant circles make up a set with large Lebesque measure;
2. each invariant circle has its stable and unstable manifold;
3. it is required that the stable manifold of each circle intersects its unstable manifold transversally;
4. it is guaranteed by the modulus of continuity of weak KAM solutions in the relevant parameter.

many pieces of normally hyperbolic invariant cylinders are joined by some annulus of cohomology equivalence.

Diffusion orbits are constructed as local minimum of action-minizing orbits which shadows a sequence of Aubry sets.

1. A Hamiltonian (Lagrangian) is called Tonelli if
   1. its Hessian in the action variable is positive definite $\partial^2_{yy} H > 0$;
   2. superlinear growth $\frac{H}{|y|} \to \infty$ as $|y| \to \infty$;
   3. each orbit can be extended to the whole $\mathbb{R}$

   Let
   $$L(x, \dot{x}, t) = \max_y(\langle \dot{x}, y\rangle - H(x, y, t)).$$
2. abuse the notation of closed 1-form η_c
   $$\eta_c = \sum_{i=1}^n c_i dx_i \qquad \Rightarrow \eta_c = \sum_{i=1}^n c_i \dot{x}_i,$$
3. modify the Lagrangian $L \to L - \eta_c$, it determines the same Euler-Lagrange equation. The closed 1-form η_c plays the role as Lagrange multiplier.

Introduction to Mather theory (continued)

1. Each a. c. curve γ_k: $[0, 2k\pi] \to \mathbb{T}^n$ with $\gamma_k(0) = \gamma_k(k)$ induces a measure μ_k on $T\mathbb{T}^n \times \mathbb{T}$ ($T\mathbb{T}^n$ for autonomous case)

$$\int f d\mu_k = \frac{1}{2k\pi} \int_0^{2k\pi} f(\gamma_k(t), \dot{\gamma}_k(t), t) dt;$$

2. The set of holonomic measure $\mathfrak{H}$ is the closure of these measures by taking $k \in \mathbb{N}$ and considering all a.c. curves;
3. for each $c \in H^1(\mathbb{T}^n, \mathbb{R})$, there exists at least one c-minimal (probability) measure $\mu_c \in \mathfrak{H}$ such that

$$\int (L - \eta_c) d\mu_c = \inf_{\nu \in \mathfrak{H}} \int (L - \eta_c) d\nu = -\alpha(c);$$

such minimal measure is invariant for the Euler-Lagrange flow $\phi_L^{t*} \mu_c = \mu_c$.

1. Mather set $\tilde{\mathcal{M}}(c) = \cup \mathrm{supp}\mu_c$, $\mathcal{M}(c) = \pi\tilde{\mathcal{M}}(c)$. One can also define Aubry set and Mañé set;
2. Mañé set $\mathcal{N}(c)$ consists of c-semi-static curves $\gamma\colon \mathbb{R} \to \mathbb{T}^n$ s.t. $\forall\ t_0 < t_1$, $\tau_1 > t_0$ with $\tau_1 = t_1 \mod 2\pi$, $\xi(t_0) = \gamma(t_0)$ and $\xi(\tau_1) = \gamma(t_1)$ we have

$$\int_{t_0}^{t_1} (L - \eta_c + \alpha(c))(\gamma(t), \dot{\gamma}(t), t)dt$$
$$\le \inf_{\xi} \int_{t_0}^{\tau_1} (L - \eta_c + \alpha(c))(\xi(t), \dot{\xi}(t), t)dt;$$

3. Aubry set $\mathcal{A}(c)$ consists of c-static curves $\Rightarrow$ can be approximated by periodic curves with action approaching to zero.