

# Explicit birational geometry of Fano & CY 3-folds / C

(partly ~~partly~~ joint w. Meng Chen)

$X$ : projective normal var.

$L$ : big Weil div.

$\hookrightarrow |mL|$  defines a rational map  
birational to its image  $\forall m \gg 0$ .

Problem (Birationality problem)

Fix a family  $\mathcal{A} = \{(X, L)\}$ ,

find  $m = m(\mathcal{A})$  s.t.  $|mL|$ : birational  $\forall m \geq m(\mathcal{A})$ .

According to MMP, 3 cases are <sup>the</sup> most interesting:

(1)  $K_X$ : nef & big,  $L = K_X$  (general type)

(2)  $-K_X$ : nef & big,  $L = -K_X$  (w. Fano)

(3)  $K_X \equiv 0$ ,  $L$ : nef & big (CY)

## Boundedness results

(1) Thm (Hacon-McKernan<sup>'06</sup>, Takayama<sup>'06</sup>, Tsuji<sup>'66</sup>, Alexeev<sup>'94</sup>, H-M-Xu<sup>'14</sup>)

$X$ : general type of dim  $n$ .  
with LC singularities

$\Rightarrow \exists m = m(n)$ , s.t.  $|mK_X|$ : birational  $\forall m \geq m(n)$ .

(2) Thm (Kollar-Miyazaki-Mori<sup>'92</sup>, Kawamata<sup>'92</sup>, KMM-Takagi<sup>'60</sup>, Birkar<sup>'16</sup>)

$X$ : weak Fano of dim  $n$ .

with canonical singularities (or,  $\exists$  LC sing. for  $\epsilon > 0$ )

$\Rightarrow \exists m = m(n)$ , s.t.  $|mK_X|$ : birational  $\forall m \geq m(n)$ .

(3) if  $L$ : Cartier.  $\Rightarrow \exists m(n)$  s.t.  $|mL|$ : birat  $\forall m \geq m(n)$

(Kollar's effective bpf thm)

if  $L$ : Weil &

$X$ : singular  $\Rightarrow$  ? No results.

(maybe related to boundedness of sing of  $X$ )

Effective results in lower dimensions

	$K_X = L$ : nef & big	$-K_X = L$ : nef & big	$K_X \equiv 0$ : L: nef & big
X: sm curve	$\forall m \geq 3$	$\forall m \geq 1$	$\forall m \geq 3$
X: sm. surface	$\forall m \geq 5$ [Bombieri]	$\forall m \geq 3$	$\forall m \geq 3$ [Reider]
X: sm 3-folds (or. Gorenstein canonical sing)	$\forall m \geq 5$ [Chen-Chen-Zhang '07]	$m \geq 4$ [Fukuda '91]	$\forall m \geq 5$ [Oguiso '91] Oguiso-Peterzell '91
X: terminal 3-folds (canonical)	$m \geq 61$ [Chen-Chen '15] ( $m \geq 27??$ )	? [Chen-J] ( $m \geq 33??$ )	? [J] (??)

Main Thm

X: normal projective var  
with  $\mathbb{Q}$ -factorial terminal singularities  
(or canonical sing)

X: weak  $\mathbb{Q}$ -Fano  $\iff -K_X$ : nef & big  
 $\mathbb{Q}$ -Fano  $\iff -K_X$ : ample.  $\rho(X) = 1$ .  
 $\mathbb{Q}$ -CY  $\iff K_X \equiv 0$ .

Thm 1 [Chen-J]

X: weak  $\mathbb{Q}$ -Fano 3-fold  
( $\mathbb{Q}$ -Fano)

$\implies |mK_X|$ : birational  $\forall m \geq 97$  (39)

Thm 2 [J]

X:  $\mathbb{Q}$ -CY 3-fold  
L: nef & big Weil div

$\implies |mL|$ : birational  $\forall m \geq 17$ .

Example (1)  $X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$ :  $\mathbb{Q}$ -Fano 3-fold.

$| -32K_X |$  not birat.

(2)  $X_{10} \subset \mathbb{P}(1, 1, 1, 2, 5)$ : sm. CY 3-fold

$L = \mathcal{O}(1)$   $|4L|$ : not birat.

Idea of proof:

Step 1 Find some good criterion for  $|mL|$  to be birational which is determined by some quantities (as  $h^0(mL)$ )

Step 2 Estimate those quantities to get upper bound.

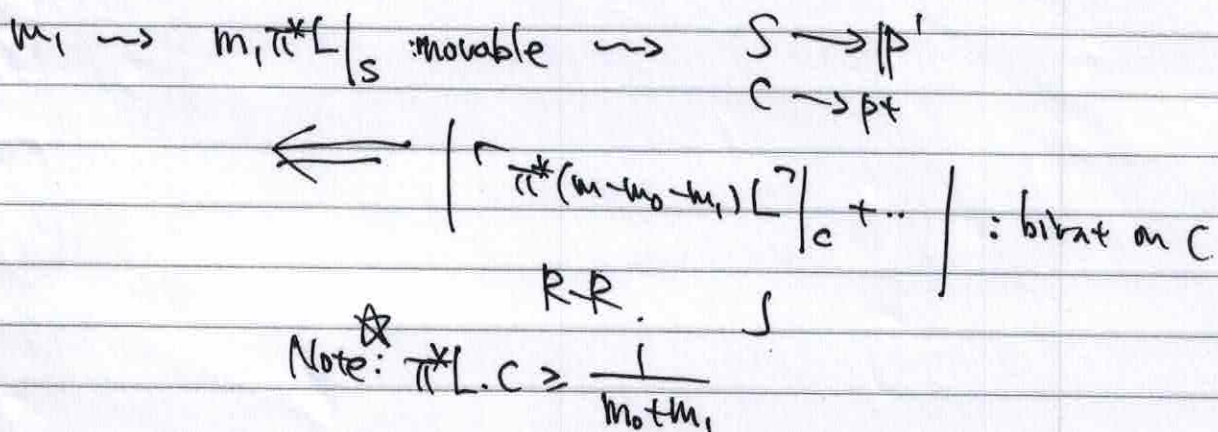
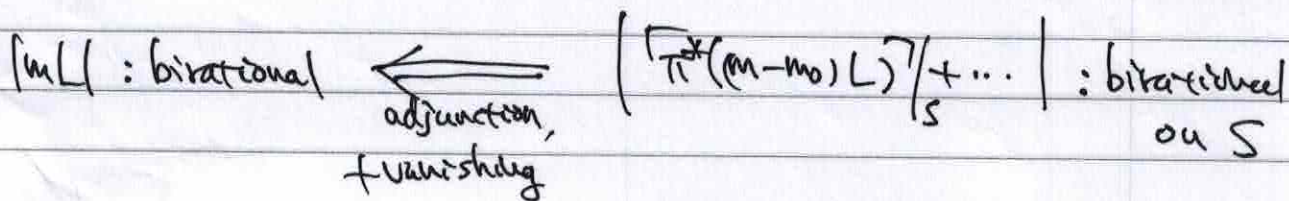
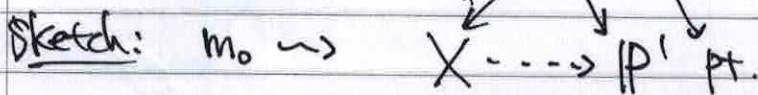
Criterion (simplified version)

- (1)  $L = -K_X$  : nef & big
- (2)  $L$  : nef big,  $K_X \equiv 0$

$m_0$  :  $h^0(m_0 L) \geq 2 \quad \forall m \geq m_0$

$m_1$  :  $\dim \text{Im } \Phi_{|m_1 L|} > 1$ . i.e.  $|m_1 L|$  is not a pencil of surfaces.

$\Rightarrow |mL|$  : birational  $\forall m \geq \begin{cases} 3m_0 + 3m_1 & \textcircled{1} \\ \frac{5}{3}(m_0 + m_1 + 1) & \textcircled{2} \end{cases}$



Examples:  $\textcircled{1} X_{66} \subset P(1, 5, 6, 22, 33)$  (Q-Fano 3-fold)

$\Rightarrow |-32K_X|$  : not birational

$m_0 = 5, m_1 = 6 \xrightarrow{\text{criterion}} |-mK_X|$  : birat  $\forall m \geq 33$

②  $X_0 \subset \mathbb{P}(1, 1, 1, 2, 5)$  CY 3-fold.

$\Rightarrow (+4L)$ : not birat.

$m_0 = m_1 = 1 \Rightarrow |mL|$ : birat  $\forall m \geq 5$ .  
Criterion.

Remaining problem: estimate  $m_0, m_1$ .

$m_0$ :  $h^0(mL) \geq 2 \quad \forall m \geq m_0$

$m_1$ :  $\dim \mathbb{P}^1 \cong \mathbb{P}^1 \rightarrow 1 \iff h^0(m_1 L) > k_X \cdot L^3 \cdot m_1 + 1$  Cor. Index

- Use Reid's Riemann-Roch formula to estimate  $m_0, m_1$ .

$h^0(mL) = \chi(mL) = R-R + \sum_{Q \in B_X} c_Q$   
Sing pts (Baskets) Correction terms.

- Singularities are finite

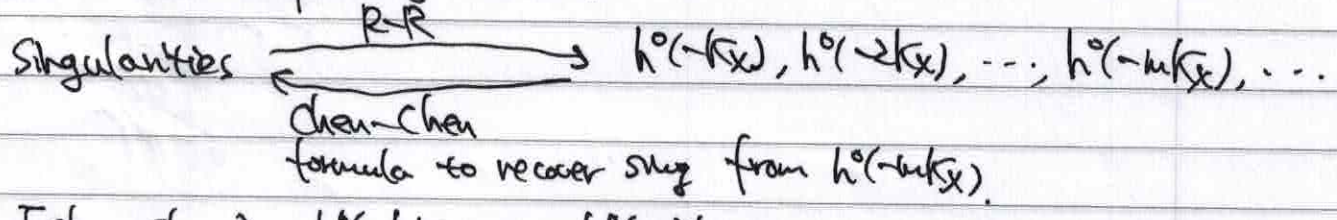
o Fano case (Kawamata, KMMT)  $\sum_i (h_i - \frac{1}{f_i}) \leq 24$   
 $\Rightarrow k_X \leq 864$   
local index of sing pts.

o CY case (Kawamata, Morrison)  $k_X \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$ .

$\rightarrow$  estimation of  $m_0$  &  $m_1$ .

For Fano case, we can do better!!

- Chen-Chen's packing method



e.g. [Chen-Chen]  $h^0(-6kX) > 0, h^0(-8kX) \geq 2$ .

Assuming  $h^0(-kX) = \dots = h^0(-6kX) = 0 \rightarrow$  NO Such Singularity  
Chen-Chen packing method

For  $m_1$ , For weak Fano 3-folds, no good method.

For  $\mathbb{Q}$ -Fano 3-folds, we can do more.  
( $\rho=1, \mathbb{Q}$ -factorial)

Recall

Thm (Alexeev)

$X$ :  $\mathbb{Q}$ -Fano 3-fold

$\Rightarrow$  general element of  $|kX|$  is prime.  
(if  $\exists$ )

Cor:

(i) if  $h^0(-kX) \geq 3 \Rightarrow \dim \text{Im } \Phi_{|kX|} > 1 \Rightarrow m_1 = 1.$

(ii) if  $h^0(-kX) = 2 \Rightarrow$  ~~h<sup>0</sup>~~  $m_1 \leq 6$   
i.e.  $|kX|$ : pencil

( $\because$  if not,  $|kX|, \dots, |6kX|$ : pencil.

$h^0 = 2, 3, \dots, 7.$

$\hookrightarrow$  NO such singularities.  
packing method

(iii) if  $h^0(-kX) = 1 \Rightarrow m_1 \leq 9.$

$\because$  if not,  $h^0(-kX), \dots, h^0(-6kX), \dots, h^0(-8kX), h^0(-9kX)$   
has finite many possibilities ( $\Leftarrow$  need  $|kX|$ : ~~the~~ prime)

e.g. 1,  $\dots$ , 1, \*, 2, 2

$\hookrightarrow$  NO such singularities.

(iv) if  $h^0(-kX) = 0$ . Alexeev's result says nothing.

$\hookrightarrow$  generalize this result to  $|mX|$ .

e.g. if  $|2kX| \neq \emptyset \xrightarrow{\text{? conditions}} |2kX|$ : prime.

$\Downarrow$  Argue as above.

$\hookrightarrow m_1 \leq 1.$