Subject: Date: 17.1.11

Explicit birational geometry of Fano & CY 3-folds / $\mathbb{C}$

(partly ~~partly~~ joint w. Meng Chen)

X: projective normal var.

L: big Weil div.

$\rightsquigarrow$ $|mL|$ defines a rational map birational to its image $\forall m \gg 0$.

Problem (Birationality problem)

Fix a family $\mathcal{A} = \{(X, L)\}$,

find $m = m(\mathcal{A})$ s.t. $|mL|$: birational $\forall m \geq m(\mathcal{A})$.

According to MMP, 3 cases are the most interesting:

(1) K_X: nef & big, $L = K_X$ (general type)

(2) $-K_X$: nef & big, $L = -K_X$ (w.Fano)

(3) $K_X \equiv 0$, L: nef & big (CY)

Boundedness results

(1) Thm (Hacon-McKernan '06, Takayama '06, Tsuji '06, H-M-Xu '14) Alexeev '94

X: general type of dim n with LC singularities

$\Rightarrow \exists m = m(n)$, s.t., $|mK_X|$: birational $\forall m \geq m(n)$.

(2) Thm (Kollár-Miyaoka-Mori '92, Kawamata '92, KMM-Takagi '00, Birkar '16)

X: weak Fano of dim n with canonical singularities (or, ε-Lc sing Fix $\varepsilon > 0$)

$\Rightarrow \exists m = m(n)$, s.t. $|-mK_X|$: birational $\forall m \geq m(n)$.

(3) if L: Cartier $\Rightarrow \exists m(n)$ s.t $|-mK|$: birat $\forall m \geq m(n)$

(Kollár's Effective bpf thm)

if L: Weil & X: singular $\Rightarrow$? No results.

(maybe related to boundedness of sing of X)

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Effective results in lower dimensions

	$K_X = L$: nef & big	$-K_X = L$: nef & big	$K_X \equiv 0$. L : nef & b
X : sm curve	✓ $m \geq 3$	✓ $m \geq 1$	✓ $m \geq 3$
X : sm. surface	✓ $m \geq 5$ [Bombieri '73]	✓ $m \geq 3$	✓ $m \geq 3$ [Reider '88]
X : sm 3-folds (or. Gorenstein canonical sing)	✓ $m \geq 5$ [Chen-Chen-Zhang '07]	$m \geq 4$ [Fukuda '91]	✓ $m \geq 5$ [Oguiso '91, Oguiso-Peternell '9]
X : terminal 3-folds (canonical)	$m \geq 61$ [Chen-Chen '15] ($m \geq 27$??)	? [Chen-J] ($m \geq 33$??)	? [J] (??)

Main Thm

X : normal projective var with $\mathbb{Q}$-factorial terminal singularities (or canonical sing)

X : weak $\mathbb{Q}$-Fano $\overset{\text{def}}{\iff}$ $-K_X$: nef & big

$\mathbb{Q}$-Fano $\overset{\text{def}}{\iff}$ $-K_X$: ample. $\rho(X) = 1$.

$\mathbb{Q}$-CY $\overset{\text{def}}{\iff}$ $K_X \equiv 0$.

Thm 1 [Chen-J]

X : weak $\mathbb{Q}$-Fano 3-fold ($\mathbb{Q}$-Fano)

$\Rightarrow |-mK_X|$: birational $\forall m \geq 97$ (39)

Thm 2 [J]

X : $\mathbb{Q}$-CY 3-fold

L : nef & big Weil div

$\Rightarrow |mL|$: birational $\forall m \geq 17$.

Example (1) $X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$: $\mathbb{Q}$-Fano 3-fold.

$|-32K_X|$ not birat.

(2) $X_{10} \subset \mathbb{P}(1, 1, 1, 2, 5)$: sm. CY 3-fold

$L = \mathcal{O}(1)$ $|4L|$: not birat.

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Idea of proof:

Step1 Find some good criterion for $|mL|$ to be birational which is determined by some quantities (as $h^0(mL)$)

Step2 Estimate those quantities to get upper bound.

Criterion (simplified version)

$$\begin{cases} L = -K_X : \text{nef \& big} & ① \\ L : \text{nef big},\ K_X \equiv 0 & ② \end{cases}$$

m_0 : $h^0(m_0 L) \geq 2$ $\forall m \geq m_0$

m_1 : $\dim \operatorname{Im} \Phi_{|m_1 L|} > 1$. ie. $|m_1 L|$ is not a pencil of surfaces.

$$\Longrightarrow |mL| : \text{birational} \quad \forall m \geq \begin{cases} 3m_0 + 3m_1 & ① \\ \frac{5}{3}(m_0 + m_1 + 1) & ② \end{cases}$$

Sketch: $m_0 \rightsquigarrow$

$$\begin{array}{ccccc} & X' & & S & \\ \swarrow^{\pi} & & \searrow & & \searrow \\ X & \dashrightarrow & \mathbb{P}^1 & & pt. \end{array}$$

$|mL|$: birational $\xLeftarrow[\text{+vanishing}]{\text{adjunction,}}$ $\left| \ulcorner \pi^*(m-m_0)L \urcorner |_S + \cdots \right|$: birational on S

$m_1 \rightsquigarrow m_1 \pi^* L|_S$ movable $\rightsquigarrow$ $S \to \mathbb{P}^1$, $C \to pt$

$\Longleftarrow \underbrace{\left| \ulcorner \pi^*(m-m_0-m_1)L \urcorner |_C + \cdots \right|}_{\text{R-R.}}$: birat on C

Note$^{☆}$: $\pi^* L . C \geq \frac{1}{m_0 + m_1}$

Examples: ① $X_{66} \subset \mathbb{P}(1,5,6,22,33)$ $\mathbb{Q}$-Fano 3-fold.

$\Rightarrow |-32K_X|$: not birational

$m_0 = 5,\ m_1 = 6 \xRightarrow[\text{criterion}]{} |-mK_X|$: birat $\forall m \geq 33$.

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② $X_{10} \subset \mathbb{P}(1,1,1,2,5)$ CY 3-fold.

$\Rightarrow |4L|$: not birat.

$m_0 = m_1 = 1 \underset{\text{Criterion}}{\Rightarrow} |mL|$: birat $\forall m \geq 5$.

<u>Remaining problem</u>: estimate m_0, m_1.

m_0 : $h^0(mL) \geq 2 \quad \forall m \geq m_0$

m_1 : $\dim \Phi_{|mL|} \geq 1 \Longleftarrow h^0(m_1 L) > r_X \cdot L^3 \cdot m_1 + 1$ ← Gor. index

- use Reid's Riemann-Roch formula to estimate m_0, m_1

$$h^0(mL) = \chi(mL) = R\text{-}R + \sum_{Q \in B_X} c_Q$$

c_Q ← correction terms. B_X ← sing pts (Baskets)

- Singularities are finite

• Fano case (Kawamata, KMMT) $\sum_i (r_i - \frac{1}{r_i}) \leq 24$ ← local index of sing pts.

$\Rightarrow r_X \leq 864$.

• CY case (Kawamata, Morrison] $r_X \in \{1,2,3,4,5,6,8,10,12\}$.

$\rightsquigarrow$ estimation of m_0 & m_1.

For Fano case, we can do <u>better</u>!!

- Chen-Chen's packing method

$$\text{Singularities} \underset{\text{Chen-Chen}}{\overset{R\text{-}R}{\rightleftarrows}} h^0(-K_X), h^0(-2K_X), \dots, h^0(-mK_X), \dots$$

formula to recover sing from $h^0(-mK_X)$.

e.g. [Chen-Chen]. $h^0(-6K_X) > 0$, $h^0(-8K_X) \geq 2$.

Assuming $h^0(-K_X) = \dots = h^0(-6K_X) = 0 \underset{\text{Chen-Chen packing method}}{\rightsquigarrow}$ No such singularities

For m_1, For weak $\mathbb{Q}$-Fano 3-folds, no good method.

For $\mathbb{Q}$-Fano 3-folds, we can do more.

($\rho = 1$, $\mathbb{Q}$-factorial)

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Recall

Thm (Alexeev)

X : $\mathbb{Q}$-Fano 3-fold

$\Rightarrow$ general element of $|-K_X|$ is prime.
(if $\exists$)

Cor:

(i) if $h^0(-K_X) \geq 3 \Rightarrow \dim \operatorname{Im} \Phi_{|-K_X|} > 1. \Rightarrow m_1 = 1.$

(ii) if $h^0(-K_X) = 2 \Rightarrow m_1 \leq 6$
i.e. $|-K_X|$: pencil

($\because$ if not. $|-K_X|, \cdots, |-6K_X|$: pencil.
$h^0 = 2, 3, \cdots, 7.$
$\xrightarrow[\text{packing method}]{}$ NO such singularities.)

(iii) if $h^0(-K_X) = 1 \Rightarrow m_1 \leq 9.$

$\because$ if not, $h^0(-K_X), \cdots, h^0(-6K_X), \cdots, h^0(-8K_X), h^0(-9K_X)$
has finite many possiblities ($\Leftarrow$ need $|-K_X|$: prime)
e.g, $1, \cdots, 1, *, 2, 2$
$\rightsquigarrow$ NO such singularities.

(iv) if $h^0(-K_X) = 0$. Alexeev's result says nothing.

$\rightsquigarrow$ generalize this result to $|-mK_X|$.

e.g. if $|-2K_X| \neq 0 \xrightarrow[+\text{?conditions}]{} |-2K_X|$: prime.
$\Downarrow$ Argue as above.

$\rightsquigarrow m_1 \leq 11.$