Dihedral Galois covers of algebraic varieties and the simple cases
(joint work with F. Catanese)

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G: finite group

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Aim

Given Y and G, determine the algebraic "building data" on Y for G-covers $\pi\colon X \to Y$, and investigate their geometry.

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G-covers
building data

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- $X = \mathrm{Spec}\,(\pi_* \mathcal{O}_X)$, and
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Assumptions

Y smooth, X normal and π flat.
(Interesting case when X smooth, then flatness of π follows.)

$\Rightarrow \quad \pi_*\mathcal{O}_X$ is a vector bundle on Y, and its fibres carry a G-action isomorphic to the regular representation of G,

$$\mathbb{C}[G] = \oplus_{g\in G}\mathbb{C}\cdot e_g\,, \quad R_h(e_g) = e_{h\cdot g}\,.$$

The canonical decomposition Let $W_1, \ldots, W_N$ be the irreducible representations of G, of degrees $n_1, \ldots, n_N$ ($n_i = \dim(W_i)$). Then there is a canonical decomposition

$$\mathbb{C}[G] \cong V_1 \oplus \ldots \oplus V_N \,,$$

where $V_i \cong W_i^{\oplus n_i}$ it is the image of the endomorphism

$$p_i = \frac{n_i}{|G|} \sum_{h \in G} \overline{\chi_i(h)} R_h \,, \qquad p_i^2 = p_i \,.$$

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$$\rightsquigarrow \qquad \pi_* \mathcal{O}_X \cong (\pi_* \mathcal{O}_X)_1 \oplus \ldots \oplus (\pi_* \mathcal{O}_X)_N$$

$(\pi_* \mathcal{O}_X)_i$ is a vector bundle on Y of rank n_i^2, the so-called "eigenbundle" corresponding to the representation W_i.

Example: Cyclic covers

$G = \langle \sigma \rangle \cong \mathbb{Z}/m\mathbb{Z}$

$$\Rightarrow \qquad N = m\,, \quad n_i = 1\,, \forall i\,, \quad (\pi_* \mathcal{O}_X)_i = \mathcal{O}_Y(L_i) \quad \text{is line bundle}$$

$\mathcal{O}_Y(L_i) = \{f \in \pi_* \mathcal{O}_X \mid \sigma^* f = \zeta^i \cdot f\}$, $\zeta \in \mathbb{C}$ is a primitive m-th root of 1.

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$$\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(L_1) \oplus \ldots \oplus \mathcal{O}_Y(L_{m-1})$$

$\mathcal{O}_Y$-algebra structure:

$$\mathcal{O}_Y(L_i) \otimes \mathcal{O}_Y(L_j) \longrightarrow \mathcal{O}_Y(L_{\overline{i+j}})\,, \quad e_i \otimes e_j \mapsto e_i \cdot e_j \tag{1}$$

Branch divisor: $\mathcal{B}_\pi = D_1 \cup \ldots \cup D_{m-1}$

D_k = union of the irreducible components $\Delta = \{\delta = 0\} \subset \mathcal{B}_\pi$, s.t. for any component $T \subset \pi^{-1}(\Delta)$, the stabilizer of the generic point of T is $\langle \sigma^k \rangle$, and $\exists t \in \mathcal{O}_{X,T}$ uniformizing parameter with

$$\mathcal{O}_{X,T} \cong \frac{\mathcal{O}_{Y,\Delta}}{\left(t^{|\sigma^k|} - \delta\right)}\,, \quad (\sigma^k)^* t = \zeta^{\frac{m}{|\sigma^k|}} \cdot t\,.$$

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$\Rightarrow$ the $\mathcal{O}_Y$-algebra structure (1) is given by:

$$L_i + L_j \equiv L_{\overline{i+j}} - \sum_{k=1}^{m-1} \epsilon_{ij}^k D_k \quad \text{in} \quad \mathrm{Pic}(Y)\,. \tag{2}$$

Where $e_i = t^{\iota_i(k)}$ where $0 \leq \iota_i(k) < |\sigma^k|$, $\epsilon_{ij}^k = \begin{cases} 0 & \text{, if } \quad \iota_i(k) + \iota_j(k) < |\sigma^k|\,, \\ 1 & \text{, otherwise}\,. \end{cases}$

Theorem

Given Y, $D_1, \ldots, D_{m-1}$ and $L_1, \ldots, L_{m-1}$ as before satisfying (2), then there exists a cyclic cover $\pi\colon X \to Y$ of order m.

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$\rightsquigarrow$ Building data for cyclic covers of degree m:

- reduced effective divisors without common components $D_1, \dots, D_{m-1} \subset Y$;
- Cartier divisors $L_1, \dots, L_{m-1} \in \mathrm{Pic}(Y)$;

such that the equation

$$L_i + L_j \equiv L_{\overline{i+j}} - \sum_{k=1}^{m-1} \epsilon_{ij}^k D_k \tag{3}$$

holds true in $\mathrm{Pic}(Y)$.

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Remark

A similar result holds true for any G abelian [Pardini, J. reine angew. Math. 417 (1991); see also Bauer-Catanese, Comment. Math. Helv. 83 (2008)].

Equations for cyclic covers:

$$\left\{ z_i \cdot z_j = \left(\prod_{k=1}^{m-1} \delta_k^{\epsilon_{ij}^k} \right) z_{\overline{i+j}} \right\} \quad \subset \mathbb{L}_1 \oplus \ldots \oplus \mathbb{L}_{m-1} \,,$$

where, $\mathbb{L}_i = \operatorname{Spec}\left(\mathcal{O}_Y(-L_i)\right)$, $D_k = \{\delta_k = 0\}$.

Remark

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The invariants of X can be computed explicitly using those of Y, the L_k's and D_k's [Pardini, J. reine angew. Math. '91].

Dihedral covers
Building data

A dihedral cover is a D_n-cover, where

$$D_n = \langle \sigma, \tau \mid \sigma^n = \tau^2 = (\sigma\tau)^2 = 1 \rangle$$

is the dihedral group of order $2n$.

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$$\pi_* \mathcal{O}_X \cong \begin{cases} \mathcal{O}_Y \oplus \mathcal{L} \oplus_{\ell=1}^{\frac{n-1}{2}} \mathcal{V}_\ell & , \quad \text{if } n \text{ is odd}\,; \\ \mathcal{O}_Y \oplus \mathcal{L} \oplus \mathcal{M} \oplus \mathcal{N} \oplus_{\ell=1}^{\frac{n}{2}-1} \mathcal{V}_\ell & , \quad \text{if } n \text{ is even}\,. \end{cases}$$

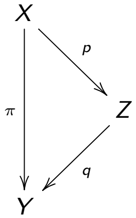
where $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are the eigenbundles of the representations of degree 1:

	σ^k	$\sigma^k\tau$
χ_1	1	1
χ_2	1	-1
χ_3	$(-1)^k$	$(-1)^k$
χ_4	$(-1)^k$	$(-1)^{k+1}$

,

The $\mathcal{V}_\ell$'s are rank 4 eigenbundles, corresponding to the representations $\rho^\ell(\sigma) = \begin{pmatrix} \zeta^\ell & 0 \\ 0 & \zeta^{-\ell} \end{pmatrix}$, $\rho^\ell(\tau) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$0 \to \mathbb{Z}/n\mathbb{Z} \to D_n \to \mathbb{Z}/2\mathbb{Z} \to 0 \quad \Rightarrow$



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$$0 \to \mathbb{Z}/n\mathbb{Z} \to D_n \to \mathbb{Z}/2\mathbb{Z} \to 0 \quad \Rightarrow \quad \begin{array}{ccc} X & \xrightarrow{p} & Z \\ & \searrow^{\pi} & \downarrow q \\ & & Y \end{array} \quad \Rightarrow \quad \pi_*\mathcal{O}_X = q_* p_* \mathcal{O}_X$$

- $p \colon X \to Z$ is a cyclic cover of degree $n \rightsquigarrow$ building data for p: $\mathcal{F}_1, \dots, \mathcal{F}_{n-1} \in \mathrm{Coh}(Z)$, reflexive of rank 1 (**divisorial sheaves**); effective Weil divisors $D_1, \dots, D_{n-1} \subset Z$ without common components, such that $\bar{\tau}^* \mathcal{F}_i \cong \mathcal{F}_{n-i}$, $\bar{\tau}(D_k) = D_{n-k}$, and (2) holds.

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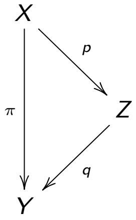

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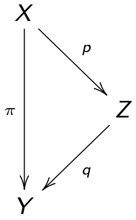

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- $\mathcal{V}_i \cong q_*\mathcal{F}_i \oplus q_*\mathcal{F}_{n-i}$.

Theorem

Let Y be a smooth variety and n be a positive integer. Then, to the following data a), b) and c), we can associate a D_n-cover $\pi\colon X \to Y$ in a natural way.

a) *A line bundle $\mathcal{L}$ and an effective reduced divisor $\mathcal{B}_q$ on Y, such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(-\mathcal{B}_q)$.*

b) *Reduced effective Weil divisors $D_1, \dots, D_{\lfloor \frac{n}{2} \rfloor}$ on $Z := \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{L})$ without common components, such that $\overline{\tau}(D_1 \cup \ldots \cup D_{\lfloor \frac{n-1}{2} \rfloor})$ doesn't have common components with $D_1 \cup \ldots \cup D_{\lfloor \frac{n-1}{2} \rfloor}$, and, in the case where n is even, $\overline{\tau}(D_{\frac{n}{2}}) = D_{\frac{n}{2}}$;*

c) *Divisorial sheaves $\mathcal{F}_1, \dots, \mathcal{F}_{\lfloor \frac{n}{2} \rfloor}$ on Z flat over $\mathcal{O}_Y$, such that (2) holds, and if n is even $\mathcal{F}_{\frac{n}{2}} = \overline{\tau}^*(\mathcal{F}_{\frac{n}{2}})$; where for $\lfloor \frac{n}{2} \rfloor < k \leq n-1$, $\mathcal{F}_k := \overline{\tau}^*(\mathcal{F}_{n-k})$ and $D_k := \overline{\tau}(D_{n-k})$, the coefficients ϵ_{ij}^k are as in (2).*

The variety X so constructed is normal if and only if, setting $\varkappa := \gcd\{k = 1, \dots, n-1 \mid D_k \neq 0\}$, then either $\varkappa = 1$, or $\frac{n}{\varkappa}\mathcal{F}_1 - \sum_{k=1}^{n-1} \frac{k}{\varkappa} D_k$ has order precisely $\varkappa$ in the group of divisorial sheaves of Z. In this case $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L} \oplus_{i=1}^{\lfloor n/2 \rfloor} q_*\mathcal{F}_i$, in particular π is flat.

Remark

1. A similar criterion for the existence of D_n-covers $\pi \colon X \to Y$ was found by Tokunaga, where X is constructed as the normalization of Y in a certain dihedral field extension of $\mathbb{C}(Y)$. Here, using the structure theorem for cyclic covers, we construct $\pi \colon X \to Y$ explicitly.

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2. For applications, one needs to understand the group of divisorial sheaves on Z.

Proposition

Sending $\mathcal{F} \mapsto q_*\mathcal{F}$ establishes a bijection between divisorial sheaves $\mathcal{F}$ on Z, flat over Y, and pairs $(\mathcal{N}, N)$ consisting of a vector bundle $\mathcal{N}$ over Y of rank 2, and a morphism $N\colon \mathcal{N} \otimes \mathcal{L} \to \mathcal{N}$, such that $N^2 = F \cdot \mathrm{Id}_{\mathcal{N}}$, where $\mathcal{B}_q = \{F = 0\}$.
Under this correspondence,

$$\begin{aligned} \mathcal{F}_1 \otimes_{\mathcal{O}_Z} \mathcal{F}_2 \quad &\leftrightarrow \quad \frac{\mathcal{N}_1 \otimes_{\mathcal{O}_Y} \mathcal{N}_2}{N_1 \otimes_{\mathcal{O}_Y} \mathrm{Id}_{\mathcal{N}_2} - \mathrm{Id}_{\mathcal{N}_1} \otimes_{\mathcal{O}_Y}} \\ \mathcal{F}^{-1} \quad &\leftrightarrow \quad \left(\mathcal{N}^* \otimes \mathcal{L}, {}^t N\right) . \end{aligned}$$

Y: smooth variety
$\mathbb{L} \to Y$: geometric line bundle
$a \in H^0(Y, \mathbb{L}^{\otimes n})$, $\quad F \in H^0(Y, \mathbb{L}^{\otimes 2})$

$$\begin{array}{ccc} X := \{(u,v) \in \mathbb{L} \oplus \mathbb{L} \mid uv = F\,,\ u^n + v^n = 2a\} & \subset & \mathbb{L} \oplus \mathbb{L} \\ \downarrow \pi & \swarrow & \\ Y & & \end{array}$$

D_n-action: $\sigma(u,v) = (\zeta u, \zeta^{-1} v)$, $\tau(u,v) = (v,u)$.

Simple D_n-covers

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Theorem (Definition: Simple dihedral covers)

If $\{a^2 - F^n = 0\}$ is smooth on $\{F \neq 0\}$, and $\{a = 0\}$ intersects $\{F = 0\}$ transversely $\Rightarrow X$ is smooth, and $\pi\colon X \to Y$ is a D_n-cover with $\mathcal{B}_\pi = \{a^2 - F^n = 0\}$.
Moreover, if $\{F = 0\} \cap \{a = 0\} \neq \emptyset \Rightarrow X$ is irreducibel.
Such a covering $\pi\colon X \to Y$ is called a **simple D_n-covering**.

Remark

1. For generic $a \in H^0(Y, \mathbb{L}^{\otimes n})$ and $F \in H^0(Y, \mathbb{L}^{\otimes 2})$, $\{a^2 - F^n = 0\}$ is smooth on $\{F \neq 0\}$ (Bertini's theorem).

2. X is the normalization of $\{z \in \mathbb{L} \,|\, z^{2n} - 2az^n + F^n = 0\} \subset \mathbb{L}$.

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Invariants of simple D_n-covers

- $\pi_* \mathcal{O}_X \cong \oplus_{i=0}^{n-1} \left[\mathcal{O}_Y(-iL) \oplus \mathcal{O}_Y(-(n-i)L) \right]$, where $\mathbb{L} = \mathrm{Spec}\,(\mathcal{O}_Y(-L))$.
- $\omega_X = \pi^* \left(\omega_Y(nL) \right)$.
- If $\dim(X) = 2$, then $K_X^2 = 2n\,(K_Y + nL)^2$.
- $\chi(\mathcal{O}_X) = \chi(\pi_* \mathcal{O}_X)$, and if $\dim(X) = 2$, then the Riemann-Roch theorem yields the following formula:

$$\chi(\mathcal{O}_X) = 2n\chi(\mathcal{O}_Y) + \frac{1}{6} n(2n^2 + 1) L \cdot L + \frac{1}{2} n^2 L \cdot K_Y \,.$$

Definition

The space of **natural deformations** of a simple dihedral covering is the family of complete intersections of $\mathbb{L} \oplus \mathbb{L}$:

$$\begin{cases} uv - F & = 0 \\ u^n - 2a + v^n + \sum_1^{n-1}(b_i u^i + c_i v^i) + d(u^n - v^n) & = 0\,, \end{cases}$$

$$\text{where } b_i, c_i \in H^0(\mathcal{O}_Y(n-i)L), d \in H^0(\mathcal{O}_Y).$$

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Motivations

$X \subset V := \mathbb{L} \oplus \mathbb{L}$ is the complete intersection of two hypersurfaces, one in $|p^*(2L)|$, the other in $|p^*(nL)|$; here $p\colon V \to Y$ is the natural projection.

$\Rightarrow 0 \to \Theta_X \to \Theta_V \otimes \mathcal{O}_X \to N_{X|V} = \mathcal{O}_X(2L') \oplus \mathcal{O}_X(nL') \to 0$, where $L' := \pi^*(L)$

Taking the direct image under π_* yields the exact sequence:

$$0 \to \pi_*\Theta_X \to \pi_*(\Theta_V \otimes \mathcal{O}_X) \to (\mathcal{O}_Y(2L) \oplus \mathcal{O}_Y(nL)) \otimes \pi_*(\mathcal{O}_X) \to 0.$$

Passing to cohomology we obtain the Kodaira-Spencer map

$$H^0((\mathcal{O}_Y(2L) \oplus \mathcal{O}_Y(nL)) \otimes \pi_*(\mathcal{O}_X)) \to H^1(\Theta_X) \to H^1(\pi_*(\Theta_V \otimes \mathcal{O}_X))$$

Theorem

Assume that $\pi\colon X \to Y$ is a simple dihedral covering. Then all small deformations of X are natural deformations of $\pi\colon X \to Y$, provided $H^1(\pi_*(\Theta_V \otimes \mathcal{O}_X)) = 0$ (that happens, for example if $H^1((\Theta_Y \oplus \mathcal{O}_Y(L)^{\oplus 2}) \otimes \pi_*(\mathcal{O}_X)) = 0$). In particular, the Kuranishi family of X is smooth, and the Kuranishi space Def(X) *is locally analytically isomorphic to*

$$\mathrm{Def}' := \mathrm{coker}\left(H^0(\pi_*(\Theta_V \otimes \mathcal{O}_X)) \to H^0((\mathcal{O}_Y(2L) \oplus \mathcal{O}_Y(nL)) \otimes \pi_*(\mathcal{O}_X))\right) .$$

Proposition (cf. Catanese, Keum, Oguiso, Math. Ann. (2003))

Let Y be a smooth variety and $L \subset Y$ be a divisor. Assume that there exist $a \in H^0(Y, \mathcal{O}_Y(nL))$ and $F \in H^0(Y, \mathcal{O}_Y(2L))$ as in the definition of simple D_n-covers.

$\Rightarrow \pi_1(Y \setminus \mathcal{B})$ admits an epimorphism onto D_n, in particular it is non-abelian, where $\mathcal{B} = \{a^2 - F^n = 0\}$.

$Y = \mathbb{P}^2$, $n = 3$, $\mathcal{O}_Y(L) = \mathcal{O}_{\mathbb{P}^2}(1)$.

$\Rightarrow a \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$, $\quad F \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, are such that the sextic curve $\{F^3 - a^2 = 0\}$ is smooth in the locus $F \neq 0$, and $\{a = 0\}$ intersects $\{F = 0\}$ transversely.

$\rightsquigarrow$ we have a smooth D_3-cover $\pi \colon X \to \mathbb{P}^2$, branched over $\mathcal{B} = \{F^3 - a^2 = 0\}$.

$\omega_X = \pi^*(\omega_{\mathbb{P}^2}(3L)) \cong \mathcal{O}_X$, $q(X) = 0$, hence X is a K3 surface. It is the CI of a quadric and a cubic in $\mathbb{P}^4$.

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$\mathcal{B}_f = \{F^3 - a^2\}$, and $\pi_1(\mathbb{P}^2 \setminus \mathcal{B})$ is the free group generated by two elements of order 2 and 3 respectively [Zariski '29].

Remark

- In general, a simple D_n-cover of $\mathbb{P}^d$, $\pi\colon X \to \mathbb{P}^d$, associated to $\mathcal{O}_Y(L) = \mathcal{O}(m)$, $F \in H^0(\mathbb{P}^d, \mathcal{O}(2m))$ and $a \in H^0(\mathbb{P}^d, \mathcal{O}(nm))$, is isomorphic to a complete intersection in the weighted projective space $\mathbb{P}^{d+2}(1, \ldots, 1, m, m)$,

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- All the small deformations of a simple D_n-cover $\pi\colon X \to \mathbb{P}^d$ associated to $\mathcal{O}_Y(L) = \mathcal{O}(m)$, $m \geq 1$, are natural deformations, if $d \geq 3$, and if $d = 2$, $(m, n) = (1, 2)$, or $m \geq 2$ and any $n \geq 2$.