Holomorphic factorization

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in SL_n(\mathbb{C})$$

$$\| $$

$$\prod_{i=1}^{N} \overset{\text{elementary matrices}}{E_i} \qquad \begin{pmatrix} 1 & & a_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 0 & & 1 \\ & & 1 \ddots 1 & \\ & 1 & & 0 \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \lambda \neq 0 & \\ & & 1 \ddots \end{pmatrix}$$

don't need

since:

(a) $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(b) $$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha^{-2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha - \alpha^2 \\ 0 & 1 \end{pmatrix}$$

$$A = L_1 U_1 \, L_2 U_2 \cdots L_n U_n$$

$$L_i = \begin{pmatrix} 1 & & 0 \\ & 1 \ddots & \\ * & & 1 \end{pmatrix} \qquad U_i = \begin{pmatrix} 1 & & * \\ & 1 \ddots & \\ 0 & & 1 \end{pmatrix}$$

$A \in SL_n(R)$ $\qquad$ R - ring

concentrate on $\quad R = \{ f : X \to \mathbb{C} \}$

X - algebraic variety $\qquad f$ - regular

topol. space $\qquad$ - continuous

complex space $\qquad$ - holomorphic

parametrized version of Gauss-elimination

0. R - euclidean e.g. $\mathbb{C}[z]$ $\qquad$ factorizes

proof $\begin{pmatrix} 1 & 0 \\ -b(z) & 1 \end{pmatrix} \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}$ $\qquad a_{21} = b \cdot a_{11} + r_1$

$$\| $$

$$\begin{pmatrix} a_{11} & * \\ r_1 & * \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \cdots \qquad \deg r_1 < \deg a_{11}$$

1. $R = \mathbb{C}[z_1, z_2]$ no factorization

$n = 2$

$$\begin{pmatrix} 1+z_1 z_2 & -z_1^2 \\ z_2^2 & 1-z_1 z_2 \end{pmatrix}$$ does not factorize

Cohn 1966

2. $R = \mathbb{C}[z_1, \dots, z_m]$ $n \geq 3$ factorizes

Suslin 1977

2* no uniform number of factors

van der Kallen 1982

General remark: If

$$A_t(x) = \prod \begin{pmatrix} 1 & & \mathbb{0} \\ & \ddots & \\ t l_i(x) & & 1 \end{pmatrix} \begin{pmatrix} 1 & & t u_i(x) \\ & \ddots & \\ \mathbb{0} & & 1 \end{pmatrix} \cdots \qquad (*)$$

$A_0(x) \equiv Id$

$l_i, u_i : X \to \mathbb{C}^{\frac{n(n-1)}{2}}$ $t \in [0,1]$

then $A : X \to Sl_n(\mathbb{C})$ is homotopic

to a constant map (null-homotopic)

4. Thurston, Vaserstein $Sl_n(\mathcal{C}(\mathbb{R}^3))$ $n \geq 3$

1986 Vaserstein $Sl_n(\mathcal{C}(\mathbb{R}))$, $Sl_n(\mathcal{C}(\mathbb{R}^2))$

Thm (Vaserstein) 1988 Calder, Siegel

5. $\exists$ universal bound $N = N(k,n)$ such that

any continuous null-homotopic $f : X \to Sl_n(\mathbb{R})$

from normal top. space X $\dim X = k < +\infty$

covering dimension

$\exists$ continuous maps $u_i, l_i : X \to \mathbb{C}^{\frac{n(n-1)}{2}}$ $i = 1, \dots, N$

with $(*)$

5. Gromov asks about holomorphic entries

1989 $SL_n(\mathcal{O}(\mathbb{C}^m))$

6. $SL_n(\mathcal{O}(X))$ X - open Riemann surface (e.g. $\mathbb{C}$)

1988 factorizes

Ramspott, Klein

<u>Thm</u> (Ivarsson, –) 2012 Ann. of Math.

$\exists$ universal number $M = M(k,n)$ such that $\forall$ all holomorphic null-homotopic maps $f : X \to SL_n(\mathbb{C})$ from a Stein space X

$\dim X = k < +\infty$

$\uparrow$

$\dim X_{reg} = k$.

$\exists$ holomorphic maps $u_i, l_i : X \to \mathbb{C}^{\frac{n(n-1)}{2}}$

with $(*)$

Stein space = complex space: holom. convex
holom. separable

smooth Stein spaces = closed analytic submanifold of $\mathbb{C}^N$

(Ivarsson, –) 2014

<u>Consequence:</u> for $n \geq 3$ and X a Stein manifold with finitely many connected components and such that all hol. $f : X \to SL_n(\mathbb{C})$ are null-homotopic Then $SL_n(\mathcal{O}(X))$ has Kazhdan's property (T)

parametric jet-interpolation of

holom. automorphisms of $\mathbb{C}^m$ $m \geq 2$

X - Stein (parameter) space

$\alpha_i : X \to J^{k,*}_{a_i(x), b_i(x)}(\mathbb{C}^m, \mathbb{C}^m)$ (invertible) $= Y$

$a_i : X \to \mathbb{C}^m$ $i = 1, \dots, N$

$a_i(x) \neq a_j(x)$ $i \neq j$

n-tuple of paramet. pt's

$b_i : Y \to \mathbb{C}^m$ $i = 1, \dots, N$

If $\alpha_1, \dots, \alpha_N : X \to Y^N_{\Delta}$ is null-homotopic

Then $\exists\, \alpha : X \to \mathrm{Aut}_{hol}(\mathbb{C}^m)$:

$$J^k_{a_i(x)}(\alpha_{x})|_k = \alpha_i \qquad i = 1, \dots, N$$

$\mathbb{C}^m \longleftrightarrow Y$ with <u>density property</u>

(Ramos Peon, Ugolini, –) 2016

<u>number of factors</u> (Ivarsson, –) 2014

a) $M(1, 2) = 2$ (2 dim. surf.) $\Rightarrow$ $M(1, n) = 2$

SL_2 (2 dim Stein X) $\forall n \geq 3$

b) $M(2, 2) = 3$ $\Rightarrow$ $M(2, n) \leq 3$

c) If $f : X \to SL_2(\mathbb{C})$ hol.

need M continuous factors,

then it needs $M+1$ hol. "

Examples <u>sharp</u>

Sketch of proof

$L_1 U_1 \dots L_M U_M$

$$
\begin{tikzcd}
E \arrow[d, "(\pi\circ f)^*"', "\Psi_M"] & \mathbb{C}^{\left(\frac{n(n-1)}{2}\cdot 2M\right)} \arrow[d, "\Psi_M"] \\
X \arrow[ur, dashed, "\tilde{f} = z_1(x),\, w_1(x), \dots"] \arrow[r, "f"] \arrow[dr, "\pi\circ f"'] & SL_n(\mathbb{C}) \arrow[d, "\pi"] \\
 & \mathbb{C}^n \setminus \{0\}
\end{tikzcd}
$$

$$z_1, w_1, \dots, z_M, w_M \mapsto \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & w_1 \\ 0 & 1 \end{pmatrix} \dots\dots \begin{pmatrix} 1 & w \\ & 1 \end{pmatrix}$$

$\boxed{z}$ $\boxed{w}$

π – proj. to last row

By Vaserstein have cont. lifting.

$\pi \circ \Psi_M \Big/ \mathbb{C}^{\dots} \setminus S_M \to \mathbb{C}^n \setminus \{0\}$ is a stratified elliptic submersion (in the sense of Gromov)

Forstneric – Gromov

$\Rightarrow$ $(\pi \circ f)^*(\dots)$ satisfies Oka principle

$\Gamma_{hol}(X, E) \subset \Gamma_{cont.}(X, E)$ is a weak homotopy equi

Problem: The Vaserstein solution may have passed through S_M

add more factors!

$$(\ ) \ \dots \ (\ ) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

induction over n-size of matrices

$$\begin{pmatrix} 1 & \\ z_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & w_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & w_2 \\ & 1 \end{pmatrix} = \Big(z_1 + z_2(z_1 w_1 + 1) ; w_1 + 1 + \dots$$

$z_1 ; z_1 w_1 + 1$ $\qquad$ $z_1 + z_2(z_1 w_1 + 1) ; z_1 w_1 + 1$

$$\boxed{z_1 + z_2(z_1 w_1 + 1) = a} \qquad (1)$$

$$w_2 \left(z_1 + z_2(z_1 w_1 + 1) \right) + z_1 w_1 + 1 = b \qquad (2)$$

<u>case 1:</u> $\quad a \neq 0$

(2) $\rightsquigarrow$ $\quad w_2 a + z_1 w_1 + 1 = b \qquad w_2 = \dfrac{b - 1 - z_1 w_1}{a}$

<u>case 2:</u> $\quad a = 0$

$$z_1 + z_2(z_1 w_1 + 1) = 0 \quad \rightsquigarrow \quad z_2 = \frac{-z_1}{b}$$

$$\boxed{z_1 w_1 + 1 = b \quad (\neq 0)}$$