K-stability and Kähler metrics, I

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Let M be a Kähler manifold. This means that M be a complex manifold together with a Kähler metric ω.

In local coordinates $z_1, \cdots, z_n$, the metric ω is given by a Hermitian positive matrix-valued function $(g_{i\bar{j}})$:

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

satisfying:

$$d\omega = 0.$$

We say ω is Kähler-Einstein if it is Kähler and Einstein, i.e.,

$$\mathrm{Ric}(\omega) \;=\; \lambda\,\omega,$$

where $\lambda = -1, 0, 1.$

In local coordinates, the Ricci curvature $\mathrm{Ric}(\omega)$ is given by

$$\mathrm{Ric}(\omega) \;=\; -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}}).$$

In 50s, E. Calabi started the study of Kähler-Einstein metrics on a compact Kähler manifold M.

The existence of Kähler-Einstein metric was established by

- Yau in 1976 when $\lambda = 0$

- Aubin, Yau independently in 1976 when $\lambda = -1$

These correspond to the cases when the first Chern class of M is zero or negative.

Now we assume $\lambda = 1$. In this case, M has positive first Chern class, i.e., M is a Fano manifold.

Not every Fano manifold admits a Kähler-Einstein metric.

There are obstructions to the existence due to 1. Matsushima in 1957, Futaki in 1983 and Tian in 1996.

• A Fano surface M has a Kähler-Einstein metric if and only if Futaki invariant vanishes (Tian, 1989).

• In 1996, I introduced the notion of K-stability and proved that if M has no non-trivial holomorphic fields and admits a Kähler-Einstein metric only if M is K-stable.

• In 2012, I gave a proof of the following:

Theorem A Fano manifold M admits a Kähler-Einstein metric if it is K-stable.

Another proof was given by Chen-Donaldson-Sun.

A “toy” case:

There is a unique 1-dimensional Fano manifold with constant curvature 1, that is, the unit sphere S^2 in $\mathbb{R}^3$!

This follows from the classical Uniformization Theorem or complex analysis.

Conic spherical structure: Given points $p_1, \cdots, p_k$ in S^2 ($k \geq 1$) and $\beta_1, \cdots, \beta_k \in (0,1)$, is there a Riemannian metric g on $S^2 \backslash \{p_i\}$ of constant curvature 1 such that g extends across each p_i as a conic metric with angle $2\pi\beta_i$?

Troyanov, McOwen, Thurston, Luo-Tian:

Such a spherical structure exists if and only if

1. $\sum_{i=1}^{k} (1 - \beta_i) \ < \ 2$;

2. For each j, $\sum_{i \neq j} (1 - \beta_i) \ > \ 1 - \beta_j$.

If we choose coordinate z such that $S^2 = \mathbb{C} \cup \{\infty\}$ and $p_i \in \mathbb{C}$ with coordinate z_i, then the existence of the above conic spherical structure is reduced to solving the equation:

$$1 + \Delta u = \prod_{i=1}^{k} |z - z_i|^{-2(1-\beta_i)} e^{F-u}.$$

where F is a given function.

This is the type of equations I studied even when I was a student at Peking University.

• Condition 1 corresponds to the Fano condition and is necessary due to the Gauss-Bonnet formula for conic metrics.

• Condition 2 simply means that the pair $(S^2, \sum(1-\beta_i)p_i)$ is K-stable as a Fano manifold S^2 together with a divisor.

Geometric Invariant theory:

Geometric invariant theory (or GIT) is a method for constructing quotients by group actions in algebraic geometry. It was developed by D. Mumford in 1965, using some ideas from a paper of Hilbert in 1893 in classical invariant theory.

Since 1970s, the GIT developed interactions with symplectic geometry, equivariant topology and differential geometry.

Let $\mathbf{G}$ be an algebraic group, say $SL(2, \mathbb{C})$ which consists of all 2×2 complex matrices with determinant 1, acting on a vector space $\mathbf{V}$. This action induces an action of $\mathbf{G}$ on the space of polynomials $R(\mathbf{V})$ on $\mathbf{V}$ by

$$\sigma \cdot f(v) = f(\sigma^{-1} v), \quad \sigma \in \mathbf{G}, v \in \mathbf{V}.$$

A polynomial f is invariant under the $\mathbf{G}$-action if $\sigma \cdot f = f$ for all $\sigma \in \mathbf{G}$. Those invariant polynomials form a commutative algebra $\mathbf{A} = R(\mathbf{V})^{\mathbf{G}}$.

For any $v \neq 0$ in $\mathbf{V}$, we say $[v] \in P(\mathbf{V})$ semi-stable if 0 is not contained in the closure of $\mathbf{G}(v)$. It follows from GIT that this is equivalent to that there is a non-constant $\mathbf{G}$-invariant homogeneous polynomial f on $\mathbf{V}$ such that $f(v) \neq 0$.

We call $[v] \in P(\mathbf{V})$ stable if $\mathbf{G}(v)$ is closed.

Denote by $P(\mathbf{V})^{ss}$ the set of all semi-stable points, then we have a "quotient" Q of $P(\mathbf{V})^{ss}$ by $\mathbf{G}$. By GIT, Q is a projective variety.

How does it relate to a Fano manifold M?

By the famous Kodaira's embedding theorem, we can embed M as a subvariety in some complex projective space $\mathbb{C}P^N$ on which the linear group $\mathbf{G} = SL(N+1, \mathbb{C})$ acts.

Using a construction of Chow, Mumford associates a nonzero vector R_M, referred as the Chow coordinate, in a vector space $\mathbf{V}$ which has an induced action by $\mathbf{G}$. Such a $\mathbf{V}$ depends only on N, $\dim M$ and the degree of $M \subset \mathbb{C}P^N$.

M is called Chow-Mumford stable if its Chow coordinate is stable.

In early 90s, I expected and tried to find a connection between the existence of Kähler-Einstein metrics and the Chow-Mumford stability. Later, my former student S. Paul also made some partial progress.

In 2000, S. Donaldson proved that the existence of Kähler-Einstein metrics implies the Chow-Mumford stability.

Now we know that the K-stability is the right condition. In fact, for many years in early 90s, I tried to relate the K-stability to the Chow-Mumford stability even though I already knew the way of defining the K-stability.

The K-stability does not fit into the classical Geometric Invariant Theory and needs its extension.

We recall **Futaki invariant**: A character of $\eta(M)$ of holomorphic vector fields on M. Let ω_0 be a Kähler metric whose Kähler class representing the first Chern class and define

$$f_M(X) = \int_M X(h_0)\,\omega_0^n,$$

where h_0 is chosen by

$$\mathrm{Ric}(\omega_0) - \omega_0 = \sqrt{-1}\partial\bar{\partial}h_0, \quad \int_M \left(e^{h_0} - 1\right)\omega_0^n = 0.$$

Futaki: f_M is an invariant and vanishes if M admits a Kähler-Einstein metric.

There are Fano manifolds with non-vanishing Futaki invariant, e.g., the blow-up of $\mathbb{C}P^2$ at one or two points.

Futaki invariant can be expressed in terms of Bott-Chern classes (Tian, 1994). This leads to a residue formula computing the invariant by using the equivariant Riemann-Roch Theorem (Futaki, Tian). Such a residue formula is analogous to Bott's residue formula for computing Chern numbers.

For long, only obstructions arise from the Lie algebra $\eta(M)$. This led to the speculation:

If a Fano manifold M has no non-zero holomorphic fields, then M admits a Kähler-Einstein metric.

In 1996, I found a counterexample to this by using the K-stability.

To introduce the K-stability, we need to generalize the Futaki invariant to singular varieties.

It was Ding and myself who first generalized the Futaki invariant to singular varieties (1992). Our construction follows a similar route as Futaki did in the smooth case, but analysis is a bit more involved. The advantage of this construction is that the generalized Futaki invariant vanishes if there is a weakly Kähler-Einstein metric on a singular variety in a suitable sense.

By a weakly Kähler-Einstein metric, we mean a Kähler-Einstein metric ω on a smooth part of a normal variety M satisfying: $\omega = \sqrt{-1}\partial\bar{\partial}\varphi$ locally for a bounded function φ and $\omega_0 \leq c\,\omega$ for some $c > 0$.

Other generalizations of Futaki invariant:

- In 2002, Donaldson gave an algebraic definition of generalized Futaki invariant which works for any polarized varieties.
- In 2008, S. Paul gave another algebraic formula of generalized Futaki invariant in terms of Chow coordinate and hyperdiscriminant.

Let $\mathbf{G}_0 = \{\sigma(t)\} \subset \mathrm{Aut}(M)$ be a $\mathbb{C}$-action on M. This can be naturally lifted to $K_M^{-1} = \Lambda^n TM$, so we have $d(\ell) = h^0(M_0, L_0^\ell)$ and the weight $w(\ell)$ of $\mathbf{G}_0$-action on $\Lambda^{\mathrm{top}} H^0(M, K_M^{-\ell})$.

By the equivariant Riemann-Roch theorem and the Riemann-Roch theorem, we have

$$w(\ell) = \sum_{i=0}^{n+1} b_i\, \ell^{n+1-i}, \quad d(\ell) = \sum_{i=0}^{n} a_i\, \ell^{n-i}.$$

Donaldson's version of the Futaki invariant is defined by

$$f_M(\mathbf{G}_0) = -2\,n!\left(b_1 - b_0 \frac{a_1}{a_0}\right).$$

If M is smooth, it follows from the equivariant index theorem that this definition coincides with Futaki's.

• In my 1997 paper, I introduced the notion of CM line bundle which can be also used to define the generalized Futaki invariant.

Let $\mathbf{G}_0 = \{\sigma(t)\}_{t\in\mathbb{C}^*}$ be an algebraic subgroup of $\mathbf{G}$ preserving a subvariety $M_0 \subset \mathbb{C}P^N$ and L_0 be the restriction of the hyperplane bundle to M_0. Then it induces an action on the CM-line which has a weight, referred as the CM-weight $\mathbf{w}_{cm}(\mathbf{G}_0)$. It was proved by Paul-Tian in 2004 that the CM-weight is the same as the generalized Futaki invariant.

The CM weight can be also identified with the first Chern class of the CM line bundle over certain compactification of $\mathbf{G}_0$ and turns out to be easier to use in algebraic geometry as Li and Xu manifested in their paper (2011).

K-stability:

By Kodaira, we can embed $M \mapsto \mathbb{C}P^N$ as a subvariety.

As above, set $\mathbf{G} = SL(N+1, \mathbb{C})$. For any algebraic subgroup $\mathbf{G}_0 = \{\sigma(t)\}_{t \in \mathbb{C}^*}$ of $\mathbf{G}$, there is a unique limiting cycle

$$M_0 = \lim_{t \to 0} \sigma(t)(M) \subset \mathbb{C}P^N.$$

One can associate a weight $\mathbf{w}(\mathbf{G}_0)$ which is either the generalized Futaki invariant as defined by Ding-Tian or Donaldson or the CM weight.

M is called semi K-stable for the embedding $M \subset \mathbb{C}P^N$ if $\mathbf{w}(\mathbf{G}_0) \geq 0$ for any $\mathbf{G}_0 \subset \mathbf{G}$.

M is called K-stable if it is semi K-stable and $\mathbf{w}(\mathbf{G}_0) > 0$ unless M_0 is biholomorphic to M.

As I said before, the K-stability is the necessary and sufficient condition for the existence of Kähler-Einstein metrics.

In the above, we confine ourselves to the case of Fano manifold. In fact, the K-stability can be also defined for any polarized projective manifold (M, L) in a similar way, where L is a positive line bundle over M.

General YTD conjecture: If (M, L) is K-stable, then M admits a Kähler metric of constant scalar curvature and with Kähler class $c_1(L)$.

K-stability and Geometric Invariant Theory:

The K-stability does not fit in the classical frame of Geometric Invariant Theory as the Chow-Mumford stability does. The GIT involves only one representation of G while the right setting for K-stability involves a pair of representations of G as manifested in the work of S. Paul. So the study of K-stability leads to an extension of GIT, say EGIT.

Extending Geometric Invariant Theory:

Let $\mathbf{V}$ and $\mathbf{W}$ be two representations of $\mathbf{G}$. Given a pair $v \in \mathbf{V} \setminus \{0\}$ and $w \in \mathbf{W} \setminus \{0\}$, we say the pair (v, w) is semistable if

$$\overline{\mathbf{G}[v, w]} \cap \overline{\mathbf{G}[0, w]} = \emptyset \quad \text{in } P(\mathbf{V} \oplus \mathbf{W}).$$

If $\mathbf{W} = \mathbb{C}, w = 1$ be the trivial 1-dimensional representation of $\mathbf{G}$. Then $(v, 1)$ is semistable if and only if 0 is not in the closure of the orbit $\mathbf{G}v$. In other words, v is semistable in the usual sense of Geometric Invariant Theory.

K-stability fits well in the frame of the extended GIT:

For each M embedded in $\mathbb{C}P^N$, Paul associates the hyperdiscriminant Δ_M and the Chow coordinate R_M. They lie in two vector spaces $\mathbf{V}$ and $\mathbf{W}$ on which $\mathbf{G}$ acts naturally.

Paul showed that M is semi K-stable is equivalent to the semistability of the pair (Δ_M, R_M).

In some sense, stability of pairs corresponds to the GIT for the representation of $\mathbf{G}$ on the difference $\mathbf{V} - \mathbf{W}$.

How is the K-stability related to the existence of Kähler-Einstein metrics on a Fano manifold? First we introduce the K-energy:

$$F_{\omega_0}(\varphi) = -\int_0^1 \int_M \dot{\varphi}_t \left(\mathrm{Ric}(\omega_t) - \omega_t\right) \wedge \omega_t^{n-1} \wedge dt,$$

where $\{\varphi_t\}$ is any path from 0 to φ in the space of Kähler metrics.

We also put

$$J_{\omega_0}(\varphi) = \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \sqrt{-1}\,\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_0^i \wedge \omega_\varphi^{n-i-1},$$

where ω_0 is a fixed Kähler metric with $[\omega_0] = 2\pi c_1(M)$.

We say that F_0 is proper if for any sequence $\{\varphi_i$ with $\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi_1 > 0$, $F_{\omega_0}(\varphi_i) \to \infty$ whenever $J_{\omega_0}(\varphi_i) \to \infty$.

Tian (90s): If M has no non-trivial holomorphic fields, then M admits a Kähler-Einstein metric if and only if F_{ω_0} is proper.

The K-stability is closely related to this properness. Let us recall a result in my PhD thesis in 1988.

By Kodaira, for $\ell >> 1$, any basis of $H^0(M, K_M^{-\ell})$ gives an embedding $\phi : M \mapsto \mathbb{C}P^N$. So we get a family of metrics

$$\mathcal{K}_\ell = \{\frac{1}{\ell}\phi^* \tau^* \omega_{FS}\}.$$

• $\bigcup \mathcal{K}_\ell$ is dense in the space of Kähler metrics on M with Kähler class $2\pi c_1(M)$.

• The K-stability simply corresponds to the properness of F_{ω_0} restricted to $\mathcal{K}_\ell$ for some sufficiently large ℓ.

Let $M \subset \mathbb{C}P^N$ by a basis of $H^0(M, K_M^{-\ell})$ for a large ℓ. For any $\sigma \in \mathbf{G} + SL(N+1, \mathbb{C})$, we have an induced metric

$$\omega_\sigma = \frac{1}{\ell}\sigma^*\omega_{FS}|_M = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\,\varphi_\sigma, \quad \int_M \varphi_\sigma\,\omega_0^n = 0.$$

The set of such metrics can be identified with the quotient of $\mathbf{G}$ by $SU(N+1)$.

For any algebraic subgroup $\mathbf{G}_0 = \{\sigma(t)\}_{t\in\mathbb{C}^*}$ of $\mathbf{G}$, there is a unique limit cycle (counted with multiplicity)

$$M_0 = \lim_{t\to 0} \sigma(t)(M) \subset \mathbb{C}P^N.$$

Note that $\mathbf{G}_0$ preserves M_0.

As $t \to 0$, we have

$$F_{\omega_0}(\varphi_{\sigma(t)}) \,=\, -\mathbf{w}'(\mathbf{G}_0)\log|t|^2 \,-\, C.$$

where $\mathbf{w}'(\mathbf{G}_0)$ is the generalized Futake invariant of a certain $\mathbf{G}_0$-equivariant branched cover M_0' of M_0.

Thus, the K-stability implies that F_{ω_0} is proper along $\{\varphi_{\sigma(t)}\}_{t\in\mathbb{C}}$ for every algebraic one-parameter subgroup $\mathbf{G}_0$ of $\mathbf{G}$.

On the other hand, in 1996, I also introduced the CM-stability in terms of the induced action of **G** on certain CM line bundle. This CM-stability is equivalent to the properness of F_{ω_0} on $\mathcal{K}_\ell$.

In view of the Hilbert-Mumford criterion in the Geometric Invariant Theory, the K-stability implies the CM-stability.

Indeed, this is true due to me and S. Paul.

Finally, the crucial technique for proving the existence of Kähler-Einstein metrics is to establish the partial C^0-estimate I proposed in 90s.