K-stability and Kähler metricss, II

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In this talk, we will discuss the partial C^0-estimate, a technique which played a crucial role in recent progress in Kähler geometry, and some of its applications.

Assume that $L \mapsto M$ be an ample line bundle over a compact manifold.

By the Kodaira Theorem, for $\ell >> 1$, any basis of $H^0(M, L^\ell)$ gives an embedding $\phi : M \mapsto \mathbb{C}P^N$.

This theorem was refined in late 80s.

Any given Kähler metric ω with Kähler class $c_1(L)$ induces an inner product on each $H^0(M, L^\ell)$, let $\{S_i\}_{0 \le i \le N}$ be an orthonormal basis of $H^0(M, L^\ell)$ w.r.t. this inner product. Put

$$\rho_{\omega,\ell}(x) = \sum_{i=0}^{N} ||S_i||^2(x).$$

This function depends only on ω and ℓ.

By Riemann-Roch, we have

$$\int_M \rho(M)\,\omega^n = \frac{c_1(L)^n\,\ell^n}{n!}.$$

It was proved in late 80s:

$$\lim_{\ell\to\infty} \ell^{-n}\rho_{\omega,\ell} = \frac{c_1(L)^n}{n!}.$$

This implies the approximation result I mentioned on Tuesday.

In view of this and certain compactness envisioned, when $L = K_M^{-1}$, we have

Conjecture (Tian, 1990): There are $c_k = c(k,n) > 0$ for $k \geq 1$ such that for any ω with Kähler class $c_1(M)$ and Ricci curvature $\geq t_0 > 0$,

$$\rho_{\omega,\ell}(x) \geq c_\ell$$

for a sufficiently large ℓ.

There are versions of this conjecture in more general cases, such as Kähler-Ricci flow, Kähler metrics with Ricci curvature bounded from below.

Assume $M = S^2 = \mathbb{C} \cup \{\infty\}$, then $H^0(M, K_M^{-\ell})$ can be identified with the space of polynomials $p(z)$ of degree less than or equal to 2ℓ, so $N = 2\ell + 1$. The standard metric on S^2 is

$$\omega_0 = 2\sqrt{-1}\frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}$$

and any other is of the form

$$\omega = (1 + \Delta\varphi)\omega_0,$$

where Δ is the Laplacian of ω_0.

Now

$$\frac{1}{\ell}\log\rho_{\omega,\ell} \;=\; \varphi - \frac{1}{\ell}\log\left(\sum_{i=0}^{2\ell}\frac{|p_i(z)|^2}{(1+|z|^2)^\ell}\right),$$

where $\{p_i\}$ satisfies

$$\int_{\mathbb{C}}\frac{p_i(z)\bar{p}_j(z)e^{\ell\varphi}}{(1+|z|^2)^\ell}\,\frac{2\,(1+\Delta\varphi)\,dz\wedge d\bar{z}}{(1+|z|^2)^2} \;=\; \delta_{ij}.$$

The partial C^0-estimate says:

If $-\Delta \log(1+\Delta\varphi) \geq \kappa > -1$, then $\rho_{\omega,\ell}$ has a uniform lower bound.

In fact, I expect that $\rho_{\omega,\ell}$ is uniformly asymptotic to a constant, even in higher dimensions.

The partial C^0-estimate, especially its extension to polarization L other than canonical ones, can be regarded as an effective version of the base-point free in algebraic geometry. It also provides a connection between metric geometry and complex geometry.

Let ω_i be Kähler metrics with Ricci curvature $\mathrm{Ric}(\omega_i) \geq \delta\omega_i$ ($\delta > 0$) and $[\omega_i] = 2\pi c_1(M)$. By taking a subsequence if necessary, we may assume that (M, ω_i) converge to a metric space (M_∞, d_∞) in the Gromov-Hausdorff topology. A fundamental problem in geometry is to show how regular (M_∞, d_∞) is.

On the other hand, for ℓ sufficiently large, an orthonormal basis of $H^0(M, K_M^{-\ell})$ with respect to the inner product induced by ω_i gives a canonical embedding $\phi_i : M \mapsto \mathbb{C}P^N$ modulo unitary transformations. Then we have a complex limit $\bar{M}_\infty$ which is a subvariety.

In general, the metric limit M_∞ may not coincide with the complex limit $\bar{M}_\infty$.

If the partial C^0-estimate holds, then these two limits can be shown to coincide. This can be used to improve the regularity on M_∞.

For example, if ω_i are Kähler-Einstein, then by Cheeger-Colding-Tian, (M_∞, d_∞) is a smooth Kähler-Einstein manifold outside a closed subset S_∞ of complex codimension 2.

Using the partial C^0-estimate, one can show that M_∞ is a normal variety and S_∞ is a subvariety (Tian, C. Li, Donaldson-Sun).

In fact, (M_∞, d_∞) is a Kähler-Einstein orbifold outside a subvariety of complex dimension 3 (C. Li-Tian, 2015)

The partial C^0-estimate plays a very important role in proving the YTD conjecture for Fano manifolds, i.e., if M is a K-stable Fano manifold, then M admits a Kähler-Einstein metric.

There are approaches to proving this YTD conjecture: The continuity method, Ricci flow as well as variational method. I will focus on the continuity method.

It has been known that M admits a Kähler-Einstein metric if and only if one can solve

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_0 - \varphi}\,\omega_0^n,$$

where ω_0 and h_0 are given.

One approach to establishing the existence is Aubin's continuity method:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_0 - t\varphi}\,\omega_0^n.$$

Define E to be the set of $t \in [0, 1]$ such that the above is solvable. Then $0 \in E$ by the Calabi-Yau theorem and E is open by Aubin in 1983. Using Aubin-Yau's 2nd order estimate and Calabi's 3rd estimate, one only needs to show the C^0-estimate.

Such a C^0-estimate follows from the properness of the Lagrangian:

$$F_{\omega_0}(\varphi) = J_{\omega_0}(\varphi) - \frac{1}{V}\int_M \varphi\,\omega_0^n - \log\left(\frac{1}{V}\int_M e^{h_0-\varphi}\,\omega_0^n\right),$$

where

$$J_{\omega_0}(\varphi) = \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \sqrt{-1}\,\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_0^i\wedge\omega_\varphi^{n-i-1}$$

Tian (90s): If M has no non-trivial holomorphic fields, then M admits a Kähler-Einstein metric if and only if F_{ω_0} is proper.

As we said before, the K-stability is closely related to this properness.

If the partial C^0-estimate conjecture holds, then the properness of F_{ω_0} along Aubin's continuity method follows from the K-stability. More precisely, if F_{ω_0} is not proper, we can produce a $\mathbb{C}^*$-action with non-negative generalized Futaki invariant and lead to a contradiction to the K-stability. So it suffices to solve the above conjecture. This has been done by G. Szekelyhidi for Kähler metrics in the Aubin continuity method in 2014. This leads to a proof of the main theorem by using the Aubin continuity method.

There were works on the conjecture:

• Kähler-Einstein metrics when $n = 2$ (Tian);

• Kähler-Einstein metrics when $n \geq 3$ (Donaldson-Sun, Tian);

• The conjecture in full generality and its Ricci flow version (W.S.Jiang, B. Wang, R. Bamler etc.).

However, the first proof used a continuity method by deforming conic Kähler-Einstein metrics. This method is Li-Sun's modification of the continuity suggested by Donaldson.

Let M be a Kähler manifold and $D \subset M$ be a smooth divisor

A conic Kähler metric on M with angle $2\pi\beta$ $(0 < \beta \leq 1)$ along D is a Kähler metric on $M \backslash D$ asymptotically equivalent along D to the model

$$\omega_{0,\beta} = \sqrt{-1}\left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2-2\beta}} + \sum_{j=2}^{n} dz_j \wedge d\bar{z}_j\right),$$

where $z_1, z_2, \cdots, z_n$ are holomorphic coordinates such that $D = \{z_1 = 0\}$ locally.

A conic Kähler-Einstein metric is a conic Kähler metric which are also Einstein outside conic points. It satisfies as currents:

$$\mathrm{Ric}(\omega) = \mu\omega + (1-\beta)[D]$$

The study of higher dimensional conic Kähler-Einstein metrics was first proposed by myself in 1994 in order to give an effective bound on the degree of rational curves in surfaces of general type. In 2010/2011, due to the works of Brendle, Jeffres-Mazzeo-Rubinstein, Berman etc., one established a theory on existence of conic Kähler-Einstein metrics which generalizes what we had known for smooth Kähler-Einstein metrics.

Define E to be the set of $\beta \in (1-\lambda^{-1}, 1]$ such that there is a conic Kähler metric with angle $2\pi\beta$ along D ($\lambda \geq 1$).

• $\beta \in E$ if β is close to $1-\lambda^{-1}$, so E is non-empty (Jeffres-Mazzeo-Rubinstein, Berman);

• E is open (Donaldson). The openness is crucial for having a valid continuity method;

• $\beta \in E$ if the twisted Lagrangian $F_{\omega_0,\beta}$ or K-energy for β is proper (Jeffres-Mazzeo-Rubinstein)

The continuity method through conic metrics can be approximated by Aubin's continuity method, so we avoid using the linear theory for conic metrics. It has an advantage that we can allow D to be a normal-crossing divisor.

By the same arguments, one can extend the Futaki invariant to log-Futaki invariant $f_{M,\beta}$ in the conic case. Then one has a generalized K-stability for the underlying space $(M,(1-\beta)D)$ for conic Kähler metrics with conic angle $2\pi\beta$. As in the smooth case, it corresponds to the properness of the twisted functional $F_{\omega_0,\beta}$ or K-energy restricted to $\mathcal{K}_\ell$ for some sufficiently large ℓ.

Furthermore, it was observed before that if M is K-stability, so does $(M,(1-\beta)D)$ for any $\beta \in (1-\lambda^{-1},1)$.

This reduces to establishing the partial C^0-estimate for conic Kähler-Einstein metrics.

Any ω with Kähler class $c_1(M)$ induces an inner product on each $H^0(M, K_M^{-\ell})$, let $\{S_i\}_{0\leq i\leq N}$ be an orthonormal basis of $H^0(M, K_M^{-\ell})$ w.r.t. this inner product. Put

$$\rho_{\omega,\ell}(x) = \sum_{i=0}^{N} ||S_i||^2(x),$$

where $||\cdot||$ is a Hermitian norm on $K_M^{-\ell}$ with curvature $\ell\omega$.

This definition is still valid for conic Kähler metrics since corresponding Hermitian norm $||\cdot||$ is Hölder continuous.

Conic version of the partial C^0-estimate:

There are $c_k = c(k, n, \beta_0) > 0$ for $k \geq 1$ such that for any conic Kähler-Einstein metric ω in E with cone angle $2\pi\beta$ along D ($\beta \geq \beta_0 > 0$) and sufficiently large ℓ,

$$\rho_{\omega,\ell}(x) \geq c_\ell.$$

This is actually a special case of my partial C^0-estimate conjecture for Kähler metrics with Kähler class $c_1(M)$ and Ricci curvature positive.

The crucial technique for establishing the partial C^0-estimate is a compactness theorem of Cheeger-Colding-Tian type for conic metrics: Let ω_i be a sequence of conic Kähler-Einstein metrics with cone angle $2\pi\beta_i$ along D converging to (M_∞, d_∞) in the Gromov-Hausdorff topology, then

- $\exists S \subset M_\infty$, closed and of codimension 2, s.t. $M_\infty \backslash S$ is smooth
- d_∞ is induced by a Kähler-Einstein metric ω_∞ outside S
- ω_i converges to ω_∞ in the C^∞-topology outside S.

Moreover, we can prove that if $\beta_\infty = 1$, M_∞ is smooth outside $S_0 \subset S$ of codimension at least 4.

To prove the above, we use some arguments in the proof of Cheeger-Colding-Tian, but there are some new inputs, especially, in establishing the smooth convergence of ω_i outside S.

First we prove the following theorem:

Any conic Kähler-Einstein metrics with scalar curvature $n\mu$ can be approximated by smooth Kähler metrics with Ricci curvature bounded from below by μ.

If $n = 1$, it is clear and a local problem. But the method of smoothing conic metrics in dimension 1 do not extend to higher dimensions. We solve this problem by using complex Monge-Ampere equations.

Let ω be a conic Kähler-Einstein metric on M with cone angle $2\pi\beta$ along D.

Choose a smooth Kähler metric ω_0 with $[\omega_0] = c_1(M)$. Define h_0 by

$$\mathrm{Ric}(\omega_0) - \omega_0 = \sqrt{-1}\partial\bar{\partial}h_0, \quad \int_M (e^{h_0} - 1)\omega_0^n = 0.$$

This is equivalent to

$$\mathrm{Ric}(\omega_0) = \mu\omega_0 + (1-\beta)[D] + \sqrt{-1}\partial\bar{\partial}(h_0 - (1-\beta)\log ||S||_0^2),$$

where S is a holomorphic section of $K_M^{-\lambda}$ defining D and $||\cdot||_0$ is a Hermitian norm on $K_M^{-\lambda}$ with $\lambda\omega_0$ as its curvature.

Write $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ for some smooth function φ on $M \backslash D$. Then

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_0-(1-\beta)\log||S||_0^2-\mu\varphi}\omega_0^n.$$

Note that φ is Hölder continuous.

To find smooth Kähler metrics which approximate ω, we consider

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_\delta - \mu\varphi}\omega_0^n,$$

where $h_\delta = h_0 - (1-\beta)\log(\delta + ||S||_0^2) + c_\delta$ for some c_δ determined by

$$\int_M \left(e^{h_0 - (1-\beta)\log(\delta + ||S||_0^2) - c_\delta} - 1\right) \omega_0^n = 0.$$

If φ_δ is a smooth solution, then we put

$$\omega_\delta = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\delta$$

A key observation (first pointed out by me): Its Ricci curvature greater than μ whenever $\beta < 1$ and $\delta > 0$.

Using the same arguments as in my 1997 paper, one can prove

• There is a smooth solution for the above equation for any $\delta > 0$

Moreover, we can prove

• ω_δ converge to ω in the smooth topology on $M\backslash D$ and in the Gromov-Hausdorff topology on M.

Then we can apply Cheeger-Colding and Cheeger-Colding-Tian to get

- For each $p \in M_\infty$, tangent cones $T_p M_\infty$ exist and complex

- The set S of $p \in M_\infty$ for which no tangent cone $T_p M_\infty$ is $\mathbb{R}^{2n}$ is closed and of codimension 2.

Moreover, if $\lim \beta_i < 1$, then ω_i converge to ω_∞ in the smooth topology outside S.

If $\lim \beta_i = 1$, then the standard arguments do not apply. However, by our approximation theorem, there is a sequence of smooth Kähler metrics $\tilde{\omega}_i$ with Ricci curvature bounded from below by $(1 - \beta_i)\lambda$ and converging to (M_∞, d_∞) in the Gromov-Hausdorff topology.

This sequence satisfies the conditions in my earlier work with B. Wang on almost Kähler-Einstein manifolds, so M_∞ is actually smooth outside $S_0 \subset S$ of codimension at least 4.

It remains to prove that if $\lim \beta_i = 1$, ω_i converge to ω_∞ outside S. This is done by studying the limit of defining sections σ_i of D with $||\sigma_i||_i = 1$. We need to prove

- σ_i converge to a holomorphic section σ_∞ of $H^0(M_\infty, K_{M_\infty}^{-\lambda})$ in the suitable sense

- $S = \sigma_\infty^{-1}(0)$

Then one can prove that ω_i converge to ω_∞ in the smooth topology outside S.

The extension of Cheeger-Colding-Tian implies: For any $x \in M_\infty$, there are $x_i \in M$ and $\ell_i = r_i^{-2}$ such that

• $(M, r_i^{-2}\omega_i, x_i)$ converge to a tangent cone (C_x, ω_x, o) of M_∞ at x;

• Line bundles $(K_{M_i}^{-\ell_i}, ||\cdot||_i)$ converge to $(C_x \times \mathbb{C}, e^{-\rho_x^2/2})$ restricted to smooth part

By using the L^2-estimate, for a sufficiently large $\ell = \ell_i$, we can obtain a holomorphic section σ_x of $K_{M_\infty}^{-\ell}$ which approximates a trivial section over a ball of C_x. On the other hand, as I did long ago, by using a Bochner identity, we can derive a gradient estimate on σ_x. It follows that σ_x is non-zero at x.

This yields the partial C^0-estimate for conic Kähler-Einstein metrics:

A direct consequence of the partial C^0-estimate is a finer regularity for M_∞:

M_∞ is a normal variety and S is a subvariety.

In fact, we can say more about (M_∞, S).

Since ω_∞ is conic Kähler-Einstein, one can show the vanishing of log-Futaki invariant $f_{M_\infty,\beta}$. So it suffices to produce only a special degeneration of M to M_∞. One approach is to show that the automorphism group of M_∞ is reductive. I had a direct proof of this by generalizing Matsushima's arguments. This concludes the proof of YTD conjecture for Fano manifolds.

Matsushima's arguments worked only for smooth M_∞ is smooth. When M_∞ has singularity, there are technical problems on the regularity of holomorphic vector fields near the singular set.

Let me mention briefly a more recent application of the partial C^0-estimate technique.

In 2014, G. Lanave and I proposed an approach to studying the analytic Minimal Model Program by considering:

$$\omega = \omega_0 - t\,\mathrm{Ric}(\omega),$$

where ω_0 is any given Kähler metric.

Aim: Applying this family of equations to the classification of Kähler manifolds.

1. Construct a global solution with surgery, i.e., a solution for all $t \geq 0$;

2. Show such a solution converges to a canonical Kähler metric on a certain canonical model M_{can}. Such canonical metrics include Kähler-Einstein metrics.

These are similar to what we did in the Analytic Minimal Model Program through Ricci flow. But the continuity method has its own advantage, that is, the deformed metrics have Ricci curvature bounded from below.

• G. Lanave-Tian: For any initial ω_0, there is a smooth family of solutions ω_t for the above family of equations on $M \times [0, T)$, where

$$T = \sup\{ t \,|\, [\omega_0] - t\, c_1(M) > 0 \}.$$

This is an analogue of the sharp local existence theorem for the Kähler-Ricci flow.

It is proved by solving a family of complex Monge-Ampere equations. Choose a real closed $(1,1)$ form ψ representing $c_1(M)$ and a smooth volume form Ω such that $\mathrm{Ric}(\Omega) = \psi$.

Put $\tilde{\omega}_t = \omega_0 - t\psi$. Then $\omega = \tilde{\omega}_t + t\sqrt{-1}\partial\bar{\partial}u$ satisfies (**??**) if u satisfies

$$(\tilde{\omega}_t + t\sqrt{-1}\,\partial\bar{\partial}\,u)^n = e^u\,\Omega, \tag{1}$$

This scalar equation can be solved. Though it depends on the choice of ψ, the solvability of (1) and resulting metrics $\omega(t) = \tilde{\omega}_t + t\sqrt{-1}\partial\bar{\partial}u$ are independent of the choice of ψ.

What happens to $\omega(t)$ when t tends to $T < \infty$?

Conjecture: As $t \to T < \infty$, $(M, \omega(t))$ converges to a compact metric space (M_T, d_T) in the Gromov-Hausdorff topology satisfying the following:

- M_T is a Kähler variety and there is a holomorphic fibration $\pi_T : M \mapsto M_T$;

- d_T is a "nice" Kähler metric ω_T on $M_T \backslash \mathcal{S}_T$, where $\mathcal{S}_T$ is a subvariety of M_T containing all the singular points.

- $\omega(t)$ converge to ω_T on $\pi^{-1}(M_T \backslash \mathcal{S}_T)$ in the smooth topology.

In 2016, G. Lanave, Z.L. Zhang and I solved the conjecture for any projective manifold (M, ω_0) in the non-collapsing case, i.e., $\dim M_T = \dim M$.

Note that in most cases, finite-time singularity formation is non-collapsing, e.g., it holds for all projective manifolds with Kodaira number not equal to $-\infty$.

More precisely, we proved:

If M is a projective manifold and ω_0 is rational, then

• $(M, \omega(t))$ converges in the Cheeger-Gromov topology to a compact path metric space (M_T, d_T) which is the metric completion of $(M \backslash \mathcal{S}, \omega_T)$;

• M_T has regular/singular decomposition $\mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ consists of all smooth points and $\mathcal{S}$ is a closed subset of codimension 2;

• M_T is homeomorphic to a normal projective variety with $\mathcal{S}$ corresponding to a subvariety.

A crucial ingredient in the proof is the technique of partial C^0-estimate for polarized line bundles on projective manifolds. However, unlike the previous case, the involved line bundle is not canonically related to the Kähler metric on the underlying manifold. In general, this may cause serious difficulties in finding required holomorphic peak sections. Our solution was inspired by J. Song's method for Kähler-Ricci flow.

Also our proof used Kawamata's base-point free theorem. It will be desirable to have a purely differentially geometric proof.