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Chern classes of automorphic bundles

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1. Conjectures on flat bundles

Conjecture 1 (E 1996)**.** *S smooth over $\mathbb{C}$, (E,∇) Gauß-Manin connection, i.e. $E = R^i f_* \Omega^*_{X/S}$ for $f : Y \to S$ smooth projective, then $0 = c_n(E) \in CH^n(S)_{\mathbb{Q}},\ n \geq 1$.*

Earlier conjecture

Conjecture 2 (Bloch 1977, proven by Reznikov 1994)**.** *S smooth projective over $\mathbb{C}$, (E,∇) flat connection, then*

$$0 = c_n(E) \in H^{2n}_{\mathcal{D}}(S, \mathbb{Q}(n)),\ n \geq 2.$$

Slight modification of Reznikov's argument (E-Corlette 2005) yields the same result for all $n \geq 1$ for Gauß-Manin bundles, even if S is not proper. Has cycle map $CH^n(S)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(S, \mathbb{Q}(n))$ with a huge kernel. So the 'distance' between Conjectures 1 and 2 is big. (The discrepancy with $n \geq 2$ is not serious, Gauss-Manin bundles have trivial determinant). However, one has the motivic conjectures

Conjecture 3 (Beilinson 1985)**.** *S smooth projective over a number field k. Then $CH^n(S)_{\mathbb{Q}} \to \prod_{\text{complex embeddings } \iota: k \hookrightarrow \mathbb{C}} H^{2n}_{\mathcal{D}}(S_\iota, \mathbb{Q}(n))$ is injective.*

So if one believes Conjecture 3, then Reznikov's theorem should imply

Corollary 4. *S smooth projective over a number field k, (E,∇) flat connection defined over k, then $0 = c_n(E) \in CH^n(S)_{\mathbb{Q}},\ n \geq 2$.*

Only known example where Conjecture 1 is known is:

Theorem 5 (van der Geer 1999, E-V 2002)**.** *$f : Y \to S$ is an abelian scheme, and $E = R^1 f_* \Omega^*_{X/S}$is the Gauß-Manin of weight* 1.

(In fact the theorem is true for the Deligne extension by [E-V], but we won't discuss compactifications in the lecture). In this case, f induces a map $S \to \mathcal{A}_g[n]$ (some level), the Gauß-Manin is the pull-back of the universal Gauß-Manin, which in fact descends to $\mathcal{A}_g$ (even if the universal family does not), and is defined over $\mathbb{Q}$. Thus Theorem 5 is also the only example known of both Conjecture 1 and of Conjecture 3.

2. Automorphic bundles

2.1. **Shimura varieties.** On the other hand, there is another class of important flat bundles, which encompasses the weight 1 Gauß-Manin bundles: the automorphic flat bundles on Shimura varieties.

Analytically, a Shimura variety ${}_KS$ is a symmetric space $X = G(\mathbb{R})/K$, G connected semisimple Lie group, $K \subset G(\mathbb{R})$ maximal compact subgroup, divided by $\Gamma \subset G(\mathbb{R})$ a torsion-free discrete arithmetic subgroup. So ${}_KS = \Gamma\backslash G(\mathbb{R})/K$. It is an algebraic variety defined over $\mathbb{C}$:

$$ {}_KS = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K $$

$$ K \subset G(\mathbb{A}_f) \text{ neat compact open.} $$

By a theorem of Borovoi, it is defined over a number field $E = E(G, X)$ called the reflex field, and in addition there is a canonical model ${}_KS_E$.

2.2. **Automorphic bundles.** Complex linear representations W of K yield vector bundles over ${}_KS$: ${}_K[W] = \Gamma\backslash W \times G(\mathbb{R})/K$ (diagonal action of Γ). It is algebraic defined over $\mathbb{C}$:

$$ {}_K[W] \;=\; G(\mathbb{Q})\backslash W \times X \times G(\mathbb{A}_f)/K, \text{ diagonal action of } \mathbb{G}(\mathbb{Q}). $$

By a theorem of Harris, it is defined over the reflex field E.

2.3. **Flat automorphic bundles.** If W is a complex linear representation of K (equivalently a $\bar{\mathbb{Q}}$-linear representation), which is induced from a linear representation of G, then $[W]$ is flat (has a flat connection coming from the representation). Margulis' rigidity theorem implies that the automorphic bundles which are flat all come from representations of G.

2.4. **Vanishing conjecture for automorphic bundles.** So Conjecture 3, Reznikov's theorem together with the descent theorem of Borovoi-Harris implies:

Conjecture 6. When ${}_KS$ is projective, and ${}_K[W]$ comes from a representation of G, i.e. is flat, then $0 = c_n(E) \in CH^n(S)_{\mathbb{Q}}, \; n \geq 1$.

If we are optimistic, we can extend Conjecture 6 to the case where S is not necessarily proper.

Among the flat automorphic bundles, one has the weight 1 Gauß-Manin bundles on $\mathcal{A}_g$. Theorem 5 is the **only** example known where one can verify Conjecture 6. With Ben Moonen and Michael Harris we tried to think of this conjecture, of specific further cases, so far without success.

3. Continuous ℓ-adic cohomology

3.1. **New cohomology group.** As we know (Griffiths, Beilinson), extensions of $\mathbb{Q}$-Hodge structures of width ≥ 2 vanish, so Deligne cohomology is the only cohomology available stemming from extensions of Hodge structures (Beilinson's viewpoint).

However, thinking of Conjecture 6 and more generally of Conjecture 3, if E is a number field, its cohomological dimension is 2.

One has Jannsen's **continuous** ℓ-adic cohomology $H^{2n}_{\text{cont}}(S, \mathbb{Q}_\ell(n))$ which in fact boils down to the projective limit over U of standard ℓ-adic cohomology if one replaces S by a flat model $\mathcal{S}_U$ over a nontrivial open U in the spectrum of $\mathcal{O}_E$, say $\epsilon : \mathcal{S}_U \to U$. One has the Hochshild-Serre spectral sequence (which is the same as the Leray spectral sequence)

$$E_2^{st} = H^s(U, R^t\epsilon_*\mathbb{Q}_\ell(n)) \Longrightarrow H^{s+t}(\mathcal{S}_U, \mathbb{Q}_\ell(n)) \text{ yielding}$$
$$E_2^{st} = H^s(E, H^t(S_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n)) \Longrightarrow H^{s+t}_{\text{cont}}(S, \mathbb{Q}_\ell(n)).$$

If S is projective, Deligne's criterion for degeneration of spectral sequences applies (as this is standard ℓ-adic cohomology and Deligne proved hard Lefschetz). So $H^{2n}_{\text{cont}}(S, \mathbb{Q}_\ell(n))$ is filtered

$$F^2 = H^2(E, H^{2n-2}(S_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n)) \subset F^1 \subset F^0 = H^{2n}_{\text{cont}}(S, \mathbb{Q}_\ell(n))$$
$$gr^1_F = H^1(E, H^{2n-1}(X_{\bar{\mathbb{Q}}}\mathbb{Q}_\ell(n)), \ gr^0_F = H^0(E, H^{2n}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n)).$$

[If S is not projective, $F^2 = $ image $H^2(E, H^{2n-2}(S_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))$ in $H^{2n}_{\text{cont}}(S, \mathbb{Q}_\ell(n))$, gr^1_F remains unchanged, gr^0_F lies injectively in $H^0(E, H^{2n}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n)).$]

3.2. **Subconjecture.**

Conjecture 7 (Corollary of Conjecture 3)**.** The term

$$F^2 = H^2(E, H^{2n-2}(S_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))$$

is the one which

a) has no reason to vanish;
b) yet a cycle class $\xi \in CH^n(S)_{\mathbb{Q}}$, with continuous ℓ-adic cycle class $c_\ell(\xi) \in H^{2n}_{\text{cont}}(S, \mathbb{Q}_\ell(n))$ which dies in gr^0_F and gr^1_F **should** die in gr^2_F as well.

Indeed, the classes in gr^0_F and gr^1_F are complete analogs of the class in Deligne cohomology. **Point b) yields a more modest and yet fascinating conjecture.**

4. The theorem

Theorem 8 (E-H 2016)**.** *For $_KS$ a projective Shimura variety and $_K[W]$ an automorphic bundle coming from a representation of G (i.e. flat), one has*

$$0 = c_\ell(_K[W]) \in H^{2n}_{\text{cont}}(_KS, \mathbb{Q}_\ell(n)),$$

i.e. Conjecture 7 is verified.

Proof. 1) Hecke algebra $\mathcal{H}_K$ acts semi-simply on $H^{2n}_{\text{cont}}(_KS, \mathbb{Q}_\ell(n))$.

2) The classes c_ℓ of automorphic bundles (flat or not) lie in one specific eigenspace under the ''volume' character $\mathcal{H}_K$:

$$H^{2n}_{\text{cont}}(_KS, \mathbb{Q}_\ell(n)) = H^{2n}_{\text{cont}}(_KS, \mathbb{Q}_\ell(n))_v \oplus \text{other eigenspaces}.$$

3) X, which is an analytic manifold, embedds as an analytic manifold into a flag variety

$$X \hookrightarrow X^\vee$$

defined over E, and the correspondence

$$\begin{array}{ccc} G(\mathbb{Q})\backslash X \times G(\mathbb{C}) \times G(\mathbb{A}_f)/K & \longrightarrow & X^\vee \\ \downarrow & & \\ _KS & & \end{array}$$

induces (in a complicated way)

$$\begin{array}{ccc} CH^n(X^\vee)_{\mathbb{Q}_\ell} & & \\ \downarrow{\scriptstyle \text{surj}} & & \\ H^{2n}(X^\vee, \mathbb{Q}_\ell(n)) & \xrightarrow[\text{surj}]{} & H^{2n}(_KS_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v \end{array}$$

4) Thus $H^{2n-1}(_KS_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)_v = 0$ which implies $gr^1_F = 0$.

5) And

$$H^{2n-2}(_KS_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v = \oplus \mathbb{Q}_\ell(0)[\text{alg.cycle}](1),$$

thus writing

$$gr^2_F = (gr^2_F)_v \oplus \text{rest},\ c_\ell \in (gr^2_F)_v$$

one has

$$(gr^2_F)_v = \oplus H^2(E, \mathbb{Q}_\ell(0))[\text{alg.cycle}](1).$$

One further applies Poincaré duality on $H^{2n-2}(_KS_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(n))_v$ to kill the class (Beauville's argument).

□