Wild group scheme quotient singularities

Workshop on Higher Dimensional Algebraic Geometry,
Holomorphic Dynamics and Their Interactions
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Goal : Enable unified treatment of Frobenius sandwich singularities and Artin-Schreier sandwich singularities.

partly joint work with S. Schröer

1. 2-dimensional quotient singularities
2. Wild quotient singularities
3. Yet another wild quotient singularities
4. Unifying two types of wild quotient singularities

Setting

(X, x) a singularity $\underset{def}{\Leftrightarrow}$ $X = \operatorname{Spec} R$

R is a 2-dimensional normal local ring

x corresponds to the maximal ideal

Assume:

$R/\mathfrak{m}_R \cong k = \bar{k}$, char$(k) = p \geq 0$

Especially, one may treat X as a 2-dimensional complex analytic space and (X, x) as an analytically local isomorphism class, when it is defined over $\mathbb{C}$.

(X, x) : 2-dimensional quotient singularity

$\underset{def}{\Longleftrightarrow}$

$X = \operatorname{Spec} R$ and G acts freely on $k[[X, Y]]$
in codimension 1
such that $\widehat{R} \cong k[[X, Y]]^G$.

$\Longleftrightarrow$

(X, x) is locally isomorphic to the quotient of $\mathbb{A}^2_k$
by a free action of a finite group $G \subset \operatorname{Aut} \mathbb{A}^2_k$
except the origin.

1. 2-dimensional quotient singularities

Over the complex number field $\mathbb{C}$,

Linearization Theorem(Cartan 1957)

(X, x) a Complex quotient singularity

Then the action of G is linearizable.

$$G \subset \mathbf{GL}(2, \mathbb{C})$$

May assume that G does not contain reflections, that is, G is small. (Chevalley's Theorem)

Characterization of quotient singularities

(Brieskorn, Mumford, ...)

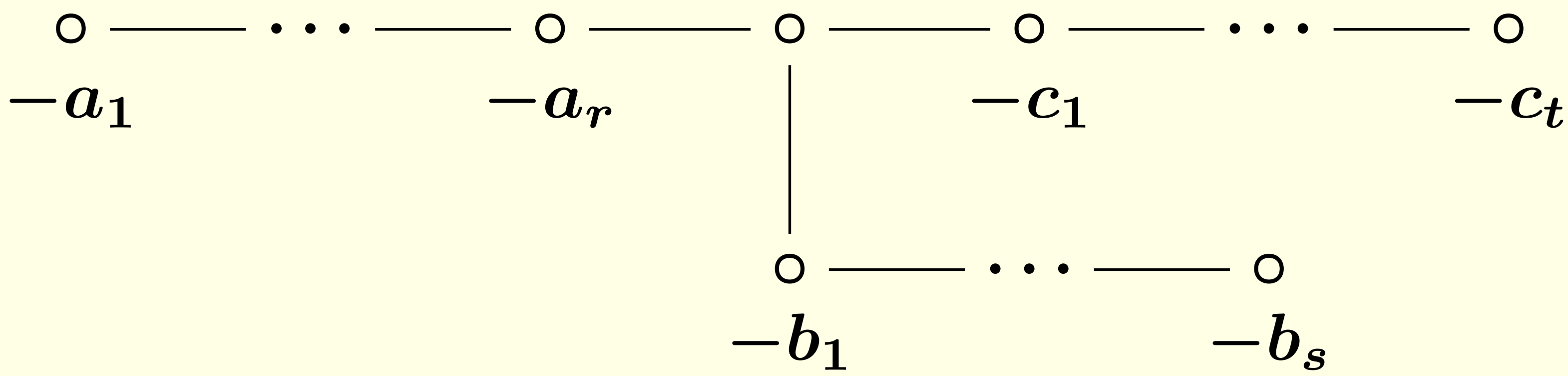
(X, x) a quotient singularity over $\mathbb{C}$

$\Longleftrightarrow$ Existence of a finite covering

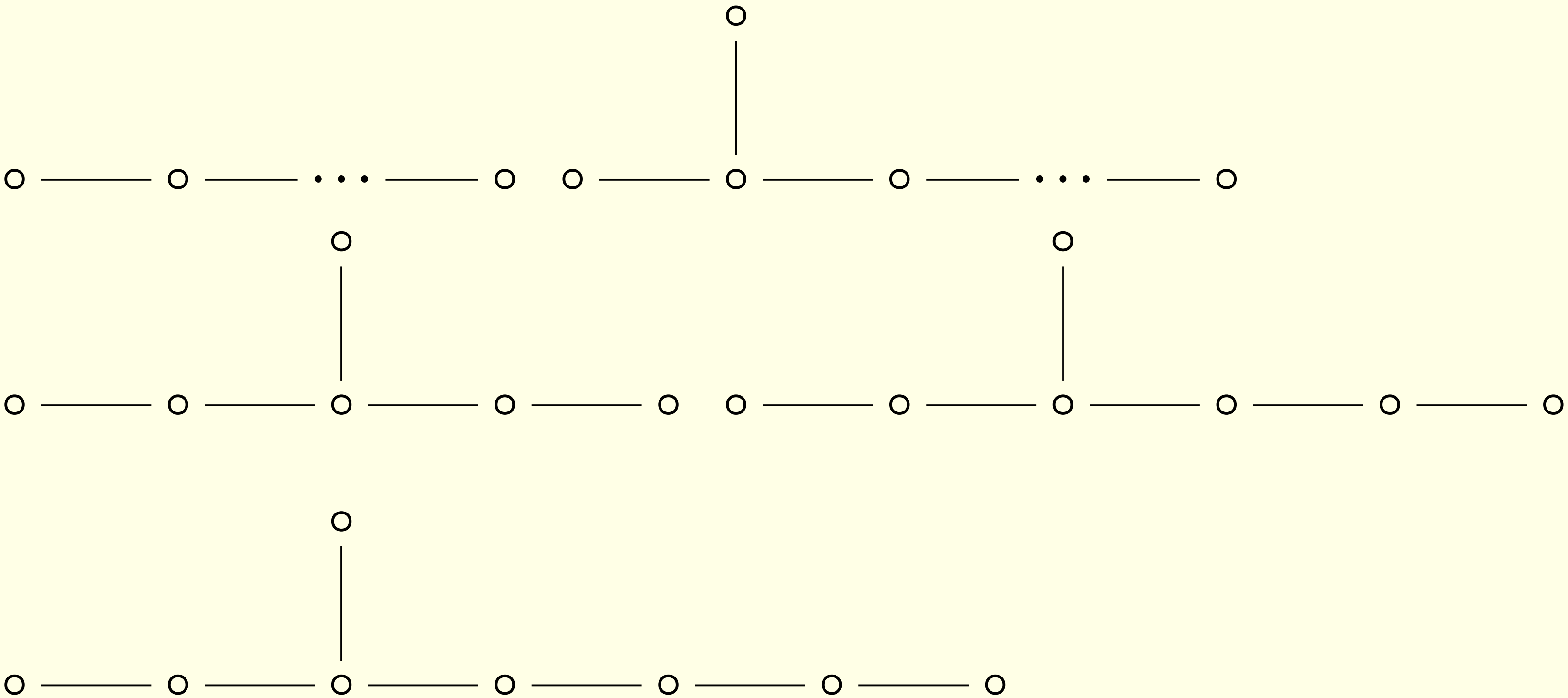
$\Longleftrightarrow$ $\pi_1^{loc}(X, x)$ is finite

$\Longleftrightarrow$ the dual graph is one of the following

$$r = 0, 1,\ s = 0, 1, 2,\ t \geq 0$$



Dual graphs of quotient singularities



the singularity is uniquely determined by above graph with appropriate weights.

RDP $/\mathbb{C}$

(X, x) : Complex RDPs

$\Longleftrightarrow$ The dual graphs are Dynkin diagrams of type A, D and E.

$\Longleftrightarrow$ Quotient singularities and Gorenstein singularities.

$\Longleftrightarrow (X, x) \cong (\mathbb{C}^2/G, 0),\ G \subset \mathbf{SL}(2, \mathbb{C})$

finite subgroup

$\Longleftrightarrow$ The hypersurface singularities in $\mathbb{C}^3$ whose defining equations are in the table.

Table: RDP $/\mathbb{C}$

group	equation	dual graph	π_1
μ_{n+1}	$z^2+x^2+y^{n+1}$ $\cong xy+z^{n+1}$	$A_n (n \geq 1)$	$\mathbb{Z}/(n+1)\mathbb{Z}$
$\widetilde{D}_{4(n-2)}$	$z^2+x^2y+y^{n-1}$	$D_n (n \geq 4)$	$\widetilde{D}_{4(n-2)}$
$\widetilde{T}$	$z^2+x^3+y^4$	E_6	$\widetilde{T}$
$\widetilde{O}$	$z^2+x^3+xy^3$	E_7	$\widetilde{O}$
$\widetilde{I}$	$z^2+x^3+y^5$	E_8	$\widetilde{I}$

μ_{n+1} : cyclic gp of order $n+1$, $\widetilde{D}_n$: binary dihedral gp of order n
$\widetilde{T}$ ($\widetilde{O}$, $\widetilde{I}$) : binary tetrahedral (octahedral, icosahedral) gp of order 24 $(48, 120)$

Tautness of RDP / $\mathbb{C}$

Isomorphism class of RDP is determined by its dual graph.

When char$(k) = p > 0$,

- the singularity is called a wild quotient singularity if p divides the order of G
- otherwise, called a tame quotient singularity.

Principle

- Tame quotient singularities behave almost same as in characteristic 0 singularities

In characteristic $p > 0$, even for the RDPs,

- some singularities have trivial π_1,
- thus, these are **NOT** quotient singularities by finite groups.
- RDP $\not\Longrightarrow$ quotient singularity in the usual sense,
- and tautness fails.

Difficulties in characteristic $p > 0$.

- Wild action cannot linearizable in general.
- Quotient singularities donot necessarily be rational any more,
  one can construct a singularity with arbitrary large genus.
- RDP $\not\Longrightarrow$ quotient singularity
- Very few examples for wild quotient singularities.

2. Wild quotient singularities

Example. Quotient singularity D_4^1 in $p = 2$

$$\sigma \in \mathbb{Z}/2\mathbb{Z} \curvearrowright A = k[[X, Y]]$$

$$\sigma(X) := \frac{X}{1+X} = X + X^2 + X^3 + \dots$$

$$= X + \frac{X^2}{1+X}$$

$$\sigma(Y) := \frac{Y}{1+Y}$$

$$A^{\langle \sigma \rangle} = k[[\frac{X^2}{1+X}, \frac{Y^2}{1+Y}, \frac{X^2Y+XY^2}{(1+X)(1+Y)}]]$$

$$\cong k[[x, y, z]]/(z^2+xyz+x^2y+xy^2)$$

Wild quotient singularities ($p = 2$, Artin 1975)

$G := \mathbb{Z}/2\mathbb{Z} \curvearrowright k[[X, Y]]$

act freely except the closed point (origin)

$\iff$

$k[[X, Y]]^G$

$= k[[X(X + a), Y(Y + b), Xb + Ya]$

$\cong k[[x, y, z]]/(z^2 + abz + xb^2 + ya^2)$

$\exists a, b \in k[[X, Y]]$

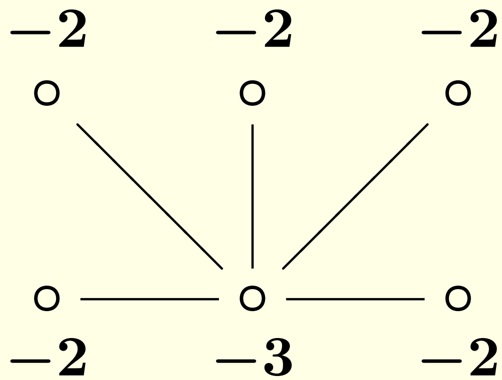
Examples by Artin's work ($p = 2$)

$$k[[X,Y]]^{\mathbb{Z}/2\mathbb{Z}}$$
$$\cong k[[x,y,z]]/(z^2 + abz + xb^2 + ya^2)$$
$$\exists a,b \in k[[U,V]]$$

$a = x, b = y \Longrightarrow D_4^1$ (RDP)

$a = y, b = x^2 \Longrightarrow E_8^2$ (RDP)

$a = x^2, b = y^2 \Rightarrow$ ⑲$_0$ (minimally elliptic)



Wild quotient singularities ($p = 3$, Peskin 1983)

$G := \mathbb{Z}/3\mathbb{Z} \curvearrowright k[[X, Y]]$

act freely except the closed point (origin)

(assume that the number of Jordan blocks of the linear part is 1)

$\Longleftrightarrow$

$k[[X, Y]]^G$

$\cong k[[x, y, z]]/(z^3 + y^{2j}z^2 - y^{3j+1} - x^2)$

$j \geq 1$

Examples by Peskin's work ($p = 3$)

$$k[[X, Y]]^{\mathbb{Z}/3\mathbb{Z}}$$
$$\cong k[[x, y, z]]/(z^3 + y^{2j} z^2 - y^{3j+1} - x^2)$$

$j = 1 \Longrightarrow E_6^1$ (RDP)

$j > 1 \Longrightarrow$ Cohen-Macaulay, non-rational

Higher dimensional case (Lorenzini-Schröer)

$G \curvearrowright k[[X_1,\dots,X_n]]$ with some assumption

$$\Longrightarrow k[[X_1,\dots,X_n]]^G \cong \frac{k[[X_1,\dots,X_n,x_1,\dots,x_n]]}{(X_i^p - a_i^{p-1}X_i - x_i \mid 1 \le i \le n)}$$

for some parameter system $a_1,\dots,a_n \in k[[x_1,\dots,x_n]]$ and G acts via $X_i \mapsto X_i + a_i$, where x_i is the norm of X_i.

RDP /char$(k) = p > 2$ (Artin 1977)

dual graph	equation	Tjurina number	π_1
$A_n (n \geq 1)$	$z^2 + x^2 + y^{k+1}$	n $(p \nmid n+1)$	$\mathbb{Z}/(n+1)\mathbb{Z}$
		$n+1$ $(p \mid n+1)$	$(\mathbb{Z}/(n+1)\mathbb{Z})'$
$D_n (n \geq 4)$	$z^2 + x^2 y + y^{k-1}$	n	$\widetilde{D}_{4(n-2)}$
E_6	E_6^0 $z^2 + x^3 + y^4$	6 $(p \neq 3)$, 9 $(p = 3)$	$\widetilde{T}$, 0
$\exists$ in $p = 3$	E_6^1 $z^2 + x^3 + y^4 + x^2 y^2$	7	$\mathbb{Z}/3\mathbb{Z}$
E_7	E_7^0 $z^2 + x^3 + xy^3$	7 $(p \neq 3)$, 9 $(p = 3)$	$\widetilde{O}$, $\mathbb{Z}/2\mathbb{Z}$
$\exists$ in $p = 3$	E_7^1 $z^2 + x^3 + xy^3 + x^2 y^2$	7	$\mathbb{Z}/6\mathbb{Z}$
E_8	E_8^0 $z^2 + x^3 + y^5$	8 $(p \neq 3, 5)$	$\widetilde{I}$
		10 $(p = 5)$	0
		12 $(p = 3)$	0
$\exists$ in $p = 3$	E_8^1 $z^2 + x^3 + y^5 + x^2 y^3$	10	0
$\exists$ in $p = 3$	E_8^2 $z^2 + x^3 + y^5 + x^2 y^2$	8	$\widetilde{T}$
$\exists$ in $p = 5$	E_8^1 $z^2 + x^3 + y^5 + xy^4$	8	$\mathbb{Z}/5\mathbb{Z}$

RDP / char$(k) = p = 2$ (Artin 1977, Lipman 1969)

dual graph	equation	Tjurina number	π_1
$A_n (n \geq 1)$	$z^{n+1} + xy$	n (n even)	$(\mathbb{Z}/(n+1)\mathbb{Z})'$
		$n+1$ (n odd)	$\mathbb{Z}/(n+1)\mathbb{Z}$
D_{2m}	D_{2m}^0 $z^2 + x^2y + xy^m$	$4m$	0
$(m \geq 2)$	D_{2m}^r $z^2 + x^2y + xy^m + xy^{m-r}z$	$4m - 2r$	0 or $D_{2(2r-m)'}$
D_{2m+1}	D_{2m+1}^0 $z^2 + x^2y + y^mz$	$4m$	0
$(m \geq 2)$	D_{2m+1}^r $z^2 + x^2y + y^mz + xy^{m-r}z$	$4m - 2r$	0 or $D_{2(4r-2m+1)}$
E_6	E_6^0 $z^2 + x^3 + y^2z$	8	$\mathbb{Z}/3\mathbb{Z}$
	E_6^1 $z^2 + x^3 + y^2z + xyz$	6	$\mathbb{Z}/6\mathbb{Z}$
E_7	E_7^0 $z^2 + x^3 + xy^3$	14	0
	E_7^1 $z^2 + x^3 + xy^3 + x^2yz$	12	0
	E_7^2 $z^2 + x^3 + xy^3 + y^3z$	10	0
	E_7^3 $z^2 + x^3 + xy^3 + xyz$	8	$\mathbb{Z}/4\mathbb{Z}$
E_8	E_8^0 $z^2 + x^3 + y^5$	16	0
	E_8^1 $z^2 + x^3 + y^5 + xy^3z$	14	0
	E_8^2 $z^2 + x^3 + y^5 + xy^2z$	12	$\mathbb{Z}/2\mathbb{Z}$
	E_8^3 $z^2 + x^3 + y^5 + y^3z$	10	0
	E_8^4 $z^2 + x^3 + y^5 + xyz$	8	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$

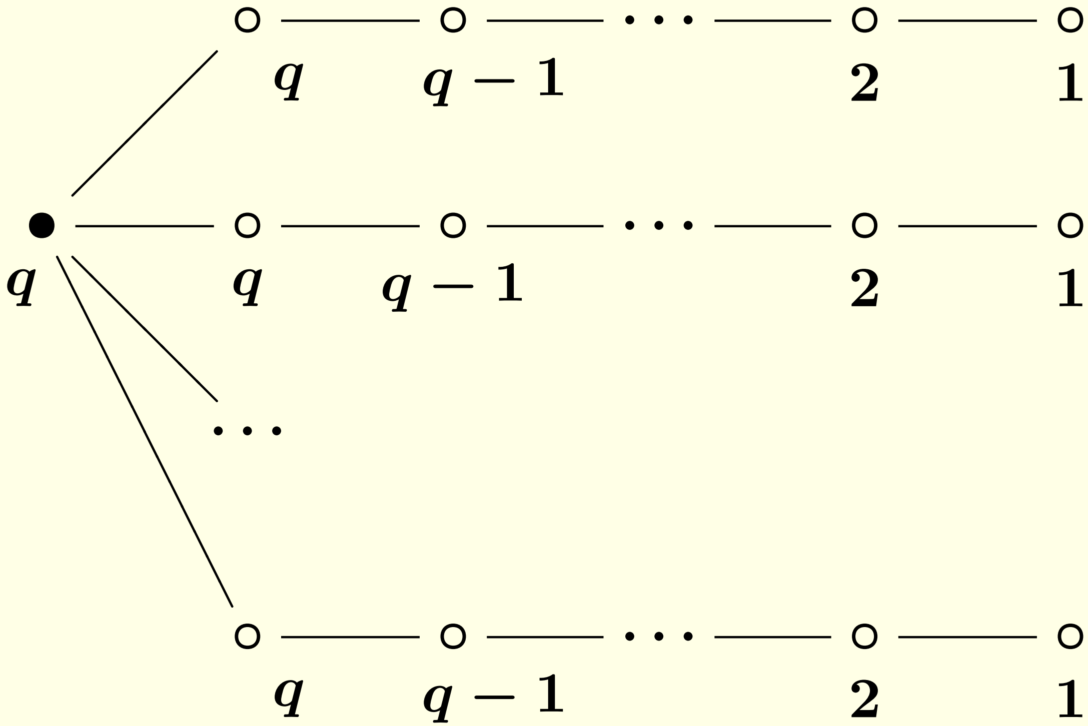
Example. (I.-Shcröer, 2012) $(IS)^1_q$

$$C : y^{q-1} = x^q - x$$

$(\mathbb{F}_q, +) \curvearrowright C \ (x \mapsto x + 1, y \mapsto y)$
induces the diagonal action on $C \times C$

Remark : This action is a generalization of the case in $q = 3$ which is the action of order 3 element in the automorphism group $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ of the supersingular elliptic curve in characteristic 3.

the dual graph of the singularity



the number of edges from ● is $q+1$

the number in the graph is the multiplicity as a divisor

the weight of edges are -2 for ○, $-q$ for ●

the fundamental genus $p_f = \frac{(q-1)(q-2)}{2}$

Singularities whose <u>dual graphs are chain</u>

$\Longleftrightarrow$ toric singularities

$\Longleftrightarrow$ Hirzebruch-Jung singularities

$\Longleftrightarrow$ "quotient" by the group scheme μ_n (any n)

(when $p \not| n$, μ_n is nothing but $\mathbb{Z}/n\mathbb{Z}$)

($\Longleftarrow$ tame cycllic quotient singularities)

wild quotient singularities

$\Longrightarrow$ tree, but the number of nodes ≥ 1,

$\exists$ sing. s.t. the number of nodes > 1.

3. Yet another wild qutient singularities

Group scheme quotient singularities

There are three specail finite flat group schemes of length p in charactersitic $p > 0$.

$$\boldsymbol{\mu_p}$$

$$\mathbb{Z}/\boldsymbol{p}\mathbb{Z}$$

$$\boldsymbol{\alpha_p}$$

There are three specail finite flat group schemes of length p in charactersitic $p > 0$.

$\boldsymbol{\mu_p}$	toric singularities
$\mathbb{Z}/\boldsymbol{p}\mathbb{Z}$	wild quotient singularities
$\boldsymbol{\alpha_p}$	

There are three specail finite flat group schemes of length p in charactersitic $p > 0$.

μ_p	toric singularities
$\mathbb{Z}/p\mathbb{Z}$	Artin-Schreier sandwich sing.
α_p	Frobenius sandwich sing.

RDP / char$(k) = p = 2$ (Artin 1977, Lipman 1969)

dual graph	equation	Tjurina number	π_1
$A_n (n \geq 1)$	$z^{n+1} + xy$	n (n even)	$(\mathbb{Z}/(n+1)\mathbb{Z})'$
		$n+1$ (n odd)	$\mathbb{Z}/(n+1)\mathbb{Z}$
D_{2m}	D^0_{2m} $z^2 + x^2y + xy^m$	$4m$	0
$(m \geq 2)$	D^r_{2m} $z^2 + x^2y + xy^m + xy^{m-r}z$	$4m - 2r$	0 or $D_{2(2r-m)'}$
D_{2m+1}	D^0_{2m+1} $z^2 + x^2y + y^m z$	$4m$	0
$(m \geq 2)$	D^r_{2m+1} $z^2 + x^2y + y^m z + xy^{m-r}z$	$4m - 2r$	0 or $D_{2(4r-2m+1)}$
E_6	E^0_6 $z^2 + x^3 + y^2 z$	8	$\mathbb{Z}/3\mathbb{Z}$
	E^1_6 $z^2 + x^3 + y^2 z + xyz$	6	$\mathbb{Z}/6\mathbb{Z}$
E_7	E^0_7 $z^2 + x^3 + xy^3$	14	0
	E^1_7 $z^2 + x^3 + xy^3 + x^2yz$	12	0
	E^2_7 $z^2 + x^3 + xy^3 + y^3 z$	10	0
	E^3_7 $z^2 + x^3 + xy^3 + xyz$	8	$\mathbb{Z}/4\mathbb{Z}$
E_8	E^0_8 $z^2 + x^3 + y^5$	16	0
	E^1_8 $z^2 + x^3 + y^5 + xy^3 z$	14	0
	E^2_8 $z^2 + x^3 + y^5 + xy^2 z$	12	$\mathbb{Z}/2\mathbb{Z}$
	E^3_8 $z^2 + x^3 + y^5 + y^3 z$	10	0
	E^4_8 $z^2 + x^3 + y^5 + xyz$	8	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$

<u>Example</u> RDP of type D_4 in $p = 2$

There are two kind of non-isomorphic RDP of type D_4 in characteristic $p = 2$, both have same Dynkin diagram as dual graphs.

- $D_4^0 \quad z^2 + x^2y + xy^2$
  Tjurina number $= 8 \quad \alpha_2$ quotient
  <u>Frobenius sandwich singularity</u>
- $D_4^1 \quad z^2 + xyz + x^2y + xy^2$
  Tjurina number $= 6 \quad \mathbb{Z}/2\mathbb{Z}$ quotient

Group scheme α_p

$$\alpha_p = \mathrm{Spec}(k[t]/t^p) = \mathrm{Spec}(k[\tau]), \quad \tau^p = 0$$

$$m : \alpha_p \times \alpha_p \to \alpha_p, \quad \epsilon : \mathrm{Spec}\, k \to \alpha_p$$

$$inv : \alpha_p \to \alpha_p$$

In the language of Hopf algebra:

$$m^* : k[\tau] \to k[\tau] \otimes k[\tau] \quad \tau \mapsto \tau \otimes 1 + 1 \otimes \tau$$

$$\epsilon^* : k[\tau] \to k \quad \tau \mapsto 0$$

$$inv^* : k[\tau] \to k[\tau] \quad \tau \mapsto -\tau$$

α_p-action on $X = \operatorname{Spec} A$

$$\sigma : \alpha_p \times X \to X$$

$$\begin{array}{ccc}
\alpha_p \times \alpha_p \times X & \xrightarrow{1_{\alpha_p} \times \sigma} & \alpha_p \times X \\
{\scriptstyle m \times 1_X}\downarrow & \circlearrowright & \downarrow{\scriptstyle \sigma} \\
\alpha_p \times \operatorname{Spec} A & \xrightarrow{\sigma} & X
\end{array}$$

$$\alpha_p \times X \underset{\epsilon \times 1_X}{\overset{\sigma}{\rightleftarrows}} \operatorname{Spec} k \times X$$

Hopf algebra side:

$$
\begin{array}{ccc}
k[\tau] \otimes k[\tau] \times A & \xleftarrow{\ 1_{k[\tau]} \otimes \sigma^* \ } & k[\tau] \otimes A \\
\uparrow {\scriptstyle m^* \otimes 1_A} & \circlearrowright & \uparrow {\scriptstyle \sigma^*} \\
k[\tau] \otimes A & \xleftarrow[\ \sigma^*\ ]{} & A
\end{array}
$$

$$
k[\tau] \otimes A \underset{\epsilon^* \otimes 1_A}{\overset{\sigma^*}{\rightleftarrows}} k \otimes A
$$

that is,

$$(1_{k[\tau]} \otimes \sigma^*) \circ \sigma^* = (m^* \otimes 1_A) \circ \sigma^*$$

$$(\epsilon^* \otimes 1_A) \circ \sigma^* = 1_A$$

$$\sigma^* : A \to k[\tau] \otimes A \cong A[\tau] \quad \tau^p = 0$$

$$x \mapsto$$

$$\delta_0(x) + \delta_1(x)\tau + \delta_2(x)\tau^2 + \cdots + \delta_{p-1}(x)\tau^{p-1}$$

$$\Longrightarrow$$

- $\delta_0 = 1_A$
- δ_1 is a k-derivation (Leibniz rule)
- $\delta_1^p = 0$(p-closed of additive type), $\delta_i = \frac{1}{i!}\delta_1^i$

α_p-action corresponds to a k-derivation

(Rudakov-Shafarevich)
$\mathrm{Ker}\,\delta_1$ ($= \delta_1$-constant) is the invariant subring of α_p-action on A. Especially, $\mathrm{Ker}\,\delta_1$ contains the Frobenius image $F(A)$ of A:

$$A \supset \mathrm{Ker}\,\delta_1 \supset F(A)$$

Example. D_4^0 singularity

$$A = k[[X, Y]]$$
$$\delta := \delta_1 := X^2 \frac{\partial}{\partial X} + Y^2 \frac{\partial}{\partial Y}$$
$$\operatorname{Ker} \delta = k[[X^2, Y^2, X^2 Y + XY^2]]$$
$$\cong k[[x, y, z]]/(z^2 + x^2 y + xy^2)$$

$$A \supset D_4^0 \text{ sing.} \supset F(A)$$

$$(F : A \to A \text{ Frobenius map})$$

$\Longrightarrow D_4^0$ is a Frobenius sandwich singularity.

Examples. $(IS)_p^0$ sing. $(p > 0)$

$$A = k[[X, Y]]$$

$$\delta := \delta_1 := X^p \frac{\partial}{\partial X} - Y^p \frac{\partial}{\partial Y}$$

$$\begin{aligned} \operatorname{Ker} \delta &= k[[X^p, Y^p, X^pY + XY^p]] \\ &\cong k[[x, y, z]]/(z^p + x^p y + x y^p) \end{aligned}$$

$$A \supset (IS)_p^0 \text{ sing.} \supset F(A)$$

$(F : A \to A$ Frobenius map$)$

$\Longrightarrow (IS)_p^0$ is a <u>Frobenius sandwich singularity</u>

4. Unifying the wild finite group quotients and finite group scheme quotients

Consider group scheme G_λ (for $\lambda \in k$) defined as the kernel of

$$\mathcal{P} := F - \lambda^{p-1} id : \mathbb{G}_a \to \mathbb{G}_a$$

that is,

$$G_\lambda = \mathbf{Spec}(k[t]/t^p - \lambda^{p-1} t) = \mathbf{Spec}(k[\tau]),$$

$$(\tau^p = \lambda^{p-1} \tau)$$

$$m : G_\lambda \times G_\lambda \to G_\lambda, \quad \epsilon : \mathbf{Spec}\, k \to G_\lambda$$

$$inv : G_\lambda \to G_\lambda$$

In the language of Hopf algebra:

$$m^* : k[\tau] \to k[\tau] \otimes k[\tau] \quad \tau \mapsto \tau \otimes 1 + 1 \otimes \tau$$

$$\epsilon^* : k[\tau] \to k \quad \tau \mapsto 0$$

$$inv^* : k[\tau] \to k[\tau] \quad \tau \mapsto -\tau$$

Then, we have

$G_0 = \alpha_p$ and $G_\lambda = \mathbb{Z}/p\mathbb{Z}$ $(\lambda \neq 0)$.

G_λ action on $X = \mathrm{Spec}\, A$:

$$\sigma_\lambda : G_\lambda \times X \to X$$

$$\begin{array}{ccc}
G_\lambda \times G_\lambda \times X & \xrightarrow{1_{G_\lambda} \times \sigma_\lambda} & G_\lambda \times X \\
\Big\downarrow {\scriptstyle m \times 1_X} & \circlearrowright & \Big\downarrow {\scriptstyle \sigma_\lambda} \\
G_\lambda \times \mathrm{Spec}\, A & \xrightarrow{\sigma_\lambda} & X
\end{array}$$

$$G_\lambda \times X \underset{\epsilon \times 1_X}{\overset{\sigma_\lambda}{\rightleftarrows}} \mathrm{Spec}\, k \times X$$

Hopf algebra side:

$$
\begin{array}{ccc}
k[\tau] \otimes k[\tau] \times A & \xleftarrow[\mathbf{1}_{k[\tau]} \otimes \sigma^*_\lambda]{} & k[\tau] \otimes A \\
\uparrow{\scriptstyle m^* \otimes \mathbf{1}_A} & \circlearrowright & \uparrow{\scriptstyle \sigma^*_\lambda} \\
k[\tau] \otimes A & \xleftarrow[\sigma^*_\lambda]{} & A
\end{array}
$$

$$
\begin{array}{ccc}
k[\tau] \otimes A & \xleftarrow{\sigma^*_\lambda} & k \otimes A \\
 & \xrightarrow[]{\epsilon^* \otimes \mathbf{1}_A} &
\end{array}
$$

that is,

$$(\mathbf{1}_{k[\tau]} \otimes \sigma^*_\lambda) \circ \sigma^*_\lambda = (m^* \otimes \mathbf{1}_A) \circ \sigma^*_\lambda$$

$$(\epsilon^* \otimes \mathbf{1}_A) \circ \sigma^*_\lambda = \mathbf{1}_A$$

Write down σ_λ^* as

$$\sigma_\lambda^* : A \to k[\tau] \otimes A \cong A[\tau] \quad \tau^p = \lambda^{p-1}\tau$$

$$x \mapsto$$

$$\delta_0^{(\lambda)}(x) + \delta_1^{(\lambda)}(x)\tau + \delta_2^{(\lambda)}(x)\tau^2 + \cdots + \delta_{p-1}^{(\lambda)}(x)\tau^p$$

Then, we have the following:

1. $\delta_0^{(\lambda)} = 1_A$
2. $\delta_1^{(\lambda)} =: \delta_\lambda$ is a k-linear homomorphism with

$$\delta_\lambda(xy) = \delta_\lambda(x)y + x\delta_\lambda(y)$$

$$+ \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} \lambda^{p-1} \delta_\lambda^i(x) \delta_\lambda^{p-i}(g)$$

3. $\delta_1^{(\lambda)} = \delta_\lambda^p = 0$ (p-closed of addtive type)
4. $\delta_i^{(\lambda)} = \frac{1}{i!}(\delta_\lambda)^i$

Especially, when $p = 2$ we have

$$\delta_\lambda(xy) = \delta_\lambda(x)y + x\delta_\lambda(y) + \underline{\lambda\delta_\lambda(x)\delta_\lambda(y)}$$

We call a k-linear homomorphism δ_λ satisfying the pseudo-Leibniz rule

$$\delta_\lambda(xy) = \delta_\lambda(x)y + x\delta_\lambda(y) + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!}\lambda^{p-1}\delta_\lambda^i(x)\delta_\lambda^{p-i}(g),$$

the pseudo-(k-)derivation.

Corollary

The action σ_λ of group scheme G_λ on A determines the pseudo-(k-)derivation δ_λ of A, and vice versa.

$$\sigma_\lambda \longleftrightarrow \delta_\lambda$$

Remark

- Invariant ring of G_λ-action = δ_λ-constant ($= \ker \delta_\lambda =: R$) $\subset A$, thus the quotient of $\mathrm{Spec}\, A$ by G_λ is $\mathrm{Spec}\, R$

- By an "exponential map", we have

$$\delta_\lambda \mapsto \varphi_\lambda \in \mathbf{Aut}(A) = \mathbf{Aut}(k[[X, Y]])$$

$$\varphi_\lambda = id + \lambda\delta_\lambda + \frac{\lambda^2}{2!}\delta_\lambda^2 + \cdots + \frac{\lambda^{p-1}}{(p-1)!}\delta_\lambda^{p-1}$$

Q. For a given k-derivation, can one <u>extend</u> a k-derivation to a pseudo-(k-)derivation ?

Q. And how ?

"YES", Idea: For a given

$$\delta = F(X,Y)\frac{\partial}{\partial X} + G(X,Y)\frac{\partial}{\partial Y},$$

Deform δ to δ_λ, substitute the norms into the Frobenius images

$$X^p \longleftrightarrow \prod_{i=0}^{p-1} \varphi_\lambda^i(X)$$

and get δ_λ, finally the action of G_λ on A, as

$$\varphi_\lambda = id + \lambda\delta_\lambda + \frac{\lambda^2}{2!}\delta_\lambda^2 + \cdots + \frac{\lambda^{p-1}}{(p-1)!}\delta_\lambda^{p-1}$$

More precisely, For a given

$$\delta = F(X,Y)\frac{\partial}{\partial X} + G(X,Y)\frac{\partial}{\partial Y}$$

"Solve" the simultaneous recursive equations:

$$(\delta_\lambda(X))^p = f\left(\prod_{i=0}^{p-1}\varphi_\lambda^i(X), \prod_{i=0}^{p-1}\varphi_\lambda^i(Y)\right)$$

$$(\delta_\lambda(Y))^p = g\left(\prod_{i=0}^{p-1}\varphi_\lambda^i(X), \prod_{i=0}^{p-1}\varphi_\lambda^i(Y)\right),$$

$$(F(X,Y)^p = f(X^p, Y^p), G(X,Y)^p = g(X^p, Y^p)).$$

If $\underline{p = 2}$, there is a good sufficient condition.

For a p-closed derivation

$$\delta = F(X, Y)\frac{\partial}{\partial X} + G(X, Y)\frac{\partial}{\partial Y},$$

if $\mathrm{ht}(F(X, Y), G(X, Y)) = 2$ (reduced),then
δ is of additive type, i.e.,$\delta^p = 0$
$\iff F(X, Y), G(X, Y) \in k[[X^p, Y^p]]$.

That is, $F(X, Y) = \exists f(X^p, Y^p)$ and
$G(X, Y) = \exists g(X^p, Y^p)$.

Therefore, if the derivation δ is reduced, then one can solve the simultaneous equations inside $k[[X, Y]]$:

$$\delta_\lambda(X) = f(X^2 + \lambda\delta_\lambda(X)X, Y^2 + \lambda\delta_\lambda(Y)Y)$$

$$\delta_\lambda(Y) = g(X^2 + \lambda\delta_\lambda(X)X, Y^2 + \lambda\delta_\lambda(Y)Y),$$

and get δ_λ, finally the action of G_λ on A as

$$\varphi_\lambda(X) = X + \lambda\delta_\lambda(X), \varphi_\lambda(Y) = Y + \lambda\delta_\lambda(Y).$$

The invariant ring

The δ_λ-constant is group scheme quotient, and is equal to the invariant ring, which is given by

$$k[[X(X+\lambda\delta_\lambda(X)), Y(Y+\lambda\delta_\lambda(Y)),$$
$$\delta_\lambda(X)Y+\delta_\lambda(Y)X]]$$

$$\cong k[[x,y,z]]/(z^2+\lambda abz+xb^2+ya^2)$$
$$\exists a,b\in k[[X,Y]]$$

Theorem($p = 2$)

Let δ be a reduced p-closed k-derivation of additive type, then one can extend δ to the p-closed pseudo-(k-)derivation of additive type δ_λ on $A = k[[X, Y]]$.

Namely, "reduced" α_p quotient singularities can be extended to G_λ-quotient singularities.

Invariant ring of G_λ-action on $A = k[[X, Y]]$ is given by

$$k[[X(X + \lambda\delta_\lambda(X)), Y(Y + \lambda\delta_\lambda(Y)), \delta_\lambda(X)Y + \delta_\lambda(Y)X]]$$

$$\cong k[[x, y, z]]/(z^2 + \lambda abz + xb^2 + ya^2)$$

$\exists a, b \in k[[X, Y]]$ non-units, relatively prime

Corollry($p = 2$) "Reduced" α_p-quotient singularities are not taut.

That is, for reduced α_p-quotient singuarltiy, there exists $\mathbb{Z}/2\mathbb{Z}$-quotient singularity whose dual graph is same as one for given α_p-quotient singularity.

Observation

$$\begin{aligned} A = k[[X, Y]] &\supset A^{\langle \varphi_\lambda \rangle} = \delta_\lambda\text{-constant} \\ &\supset \mathcal{P}(A) := k[[\mathcal{P}(X), \mathcal{P}(Y)]] \end{aligned}$$

where $\mathcal{P}$ is the map $f \mapsto f(f + \lambda\delta_\lambda(f))$.

Example. D_4 singularity in $p = 2$

$$A = k[[X, Y]],\ \delta := \delta_1 := X^2 \frac{\partial}{\partial X} + Y^2 \frac{\partial}{\partial Y}$$

$$\mathrm{Ker}\,\delta = k[[X^2, Y^2, X^2Y + XY^2]]$$
$$\cong k[[x, y, z]]/(z^2 + x^2y + xy^2)$$

Solve the simultaneous recursive equalitis:

$$\delta_\lambda(X) = f(X^2 + \lambda\delta_\lambda(X)X, Y^2 + \lambda\delta_\lambda(Y)Y)$$

$$\delta_\lambda(Y) = g(X^2 + \lambda\delta_\lambda(X)X, Y^2 + \lambda\delta_\lambda(Y)Y),$$

with $f(X, Y) = X, g(X, Y) = Y,$

then get

$$\delta_\lambda(X) = \frac{X^2}{1+\lambda X}, \quad \delta_\lambda(Y) = \frac{Y^2}{1+\lambda Y}.$$

Thus $A = k[[X, Y]] \ \supset$

$$k[[\frac{X^2}{1+\lambda X}, \frac{Y^2}{1+\lambda Y}, \delta_\lambda(X)Y + X\delta_\lambda(Y)]]$$

$$= k[[\frac{X^2}{1+\lambda X}, \frac{Y^2}{1+\lambda Y}, \frac{XY^2 + X^2Y}{(1+\lambda X)(1+\lambda Y)}]]$$

$$\supset \mathcal{P}(A)$$

Example. D_{4m} singularity in $p = 2$

$$A = k[[X, Y]],$$
$$\delta := \delta_1 := X^2 \frac{\partial}{\partial X} + Y^{2m} \frac{\partial}{\partial Y}$$
$$\operatorname{Ker} \delta = k[[X^2, Y^2, X^2Y + XY^m]]$$
$$\cong k[[x, y, z]]/(z^2 + x^2y + xy^m)$$

Solve the simultaneous recursive equalitis:

$$\delta_\lambda(X) = f(X^2 + \lambda\delta_\lambda(X)X, Y^2 + \lambda\delta_\lambda(Y)Y)$$

$$\delta_\lambda(Y) = g(X^2 + \lambda\delta_\lambda(X)X, Y^2 + \lambda\delta_\lambda(Y)Y),$$

with $f(X, Y) = X, g(X, Y) = Y^m,$

then get

$$\delta_\lambda(X) = \frac{X^2}{1+\lambda X}, \quad \delta_\lambda(Y) = \ldots \quad .$$

Thus $A = k[[X, Y]] \supset$

$$k[[\frac{X^2}{1+\lambda X}, \ldots, \delta_\lambda(X)Y + X\delta_\lambda(Y)]]$$

$$= k[[x, y, z]]/(z^2 + \lambda x y^{2m} z + x^2 y + x y^{2m})$$

$$\supset \mathcal{P}(A) = k[[\mathcal{P}(X), \mathcal{P}(Y)]]$$

Many more explicit examples.

Such as $(IS)_p$ sing. $(p > 0)$, ⑲$_0$, . . .

these are sandwiched by the regular rings $k[[X, Y]]$ and $k[[\mathcal{P}(X), \mathcal{P}(Y)]]$.

Example (Sandwiched by normal (non-regular) rings).

E_6^r singularity (sandwiched by A_2)

$$\begin{array}{ccccc}
\mathbb{A}^2 & \xrightarrow{G_\lambda=\langle\delta_\lambda\rangle} & D_4^r & \longrightarrow & \mathbb{A}^2 \\
{\scriptstyle \mu_3=\langle\tau\rangle}\downarrow & \circlearrowright & \downarrow{\scriptstyle \mu_3} & & \downarrow \\
A_2 & \xrightarrow{G_\lambda} & E_6^r & \longrightarrow & A_2
\end{array}$$

E_6^0 is a Frobenius sandwich singularity of A_2, and
E_6^1 is an Artin-Schreier sandwich singularity of A_2.

Example (Sandwiched by a singularity).

E_6^r singularity (sandwiched by A_2)

$$
\begin{array}{ccccc}
\mathbb{A}^2 & \xrightarrow{G_\lambda = \langle \delta_\lambda \rangle} & D_4^r & \longrightarrow & \mathbb{A}^2 \\
{\scriptstyle \mu_3 = \langle \tau \rangle} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \mu_3} & & \downarrow \\
A_2 & \xrightarrow{G_\lambda} & E_6^r & \longrightarrow & A_2
\end{array}
$$

E_6^0 is a Frobenius sandwich singularity of A_2, and
E_6^1 is a Artin-Schreier sandwich singularity of A_2.

E_6^r singularity (sandwiched by A_2)

$$A = k[[X, Y]]$$

$$\delta := \delta_2 := Y^2 \frac{\partial}{\partial X} + X^2 \frac{\partial}{\partial Y}$$

$$\begin{aligned} \operatorname{Ker} \delta &= k[[X^2, Y^2, X^3 + Y^3]] \\ &\cong k[[x, y, z]]/(z^2 + x^3 + y^3) \\ &\cong k[[x, y, z]]/(z^2 + x^2 y + x y^2) : D_4^0 \end{aligned}$$

$\Longrightarrow$

$$\delta_\lambda(X) = \frac{Y^2 + \lambda X^2 Y}{1 + \lambda XY},\ \delta_\lambda(Y) = \frac{X^2 + \lambda X^2 X}{1 + \lambda XY}$$

gives the G_λ action, and get D_4^r $(r = 0, 1)$

Example (Sandwiched by a singularity).

E_6^r singularity (sandwiched by A_2)

$$
\begin{array}{ccccc}
\mathbb{A}^2 & \xrightarrow{G_\lambda = \langle \delta_\lambda \rangle} & D_4^r & \longrightarrow & \mathbb{A}^2 \\
{\scriptstyle \mu_3 = \langle \tau \rangle}\downarrow & \circlearrowright & \downarrow{\scriptstyle \mu_3} & & \downarrow \\
A_2 & \xrightarrow{G_\lambda} & E_6^r & \longrightarrow & A_2
\end{array}
$$

E_6^0 is a Frobenius sandwich singularity of A_2, and
E_6^1 is a Artin-Schreier sandwich singularity of A_2.

Example. E_6^r singularity (sandwiched by A_2)

$$A = k[[X, Y]]$$

$$\tau(X) := \zeta_3 X, \quad \tau(Y) := \zeta_3^{-1} Y$$

$$A^{\langle\tau\rangle} = k[[X^3, Y^3, XY]]$$

$$\cong k[[x, y, z]]/(z^3 + xy)$$

A_2 singularity

Example. E_6^r singularity (cont.)

Note that the actions by μ_3 and δ_λ commute, thus, one can induce G_λ action on A_2 and μ_3 action on D_4^r.

$\Longrightarrow$

Get E_6^r as both a quotient of A_2 by G_λ and a quotient of D_4^r by μ_3.

<u>Example</u>. E_6^r singularity (cont.)

$$\begin{array}{ccccc} \mathbb{A}^2 & \xrightarrow{G_\lambda=\langle\delta_\lambda\rangle} & D_4^r & \longrightarrow & \mathbb{A}^2 \\ {\scriptstyle \mu_3=\langle\tau\rangle}\big\downarrow & \circlearrowright & \big\downarrow{\scriptstyle \mu_3} & & \big\downarrow \\ A_2 & \xrightarrow{G_\lambda} & E_6^r & \longrightarrow & A_2 \end{array}$$

Therefore,

E_6^0 is a <u>Frobenius sandwich singularity</u> of A_2, and
E_6^1 is a <u>Artin-Schreier sandwich singularity</u> of A_2.

Thank you,

for your attention.