FLAG VARIETIES: GEOMETRIC CHARACTERIZATION AND RIGIDITY.

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[JOINT WORK WITH GIANLUCA OCCHETTA, LUIS SOLÁ-CONDE, KIWAMU WATANABE, ANDRZEJ WEBER.]

WE WORK OVER $\mathbb{C}$

FLAG: G S-SIMPLE ALG GR $B < G$ BOREL

B MINIMAL ST. G/B PROJECTIVE

EG. $G = SL_n$ G/B "FULL FLAGS" $\{0\} = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n$

$P \subset G$ PARABOLIC $\overset{def}{\iff}$ G/P PROJECTIVE

FOR EVERY PARABOLIC $P \supset B$ (UP TO CONJ)

SO THAT $G/B \to G/P$ SURJ.

MAXIMAL PARABOLIC G/P_{max} = GRASS.

MINIMAL PARABOLIC $\neq$ BOREL $\quad G/B \xrightarrow{\mathbb{P}^1\text{-BUNDLE}} G/P_{MIN}$

EVERY ELEMENTARY FANO - MORI CONTRACTION OF G/B IS $\mathbb{P}^1$-BUNDLE.

THEOREM I X FANO MNFLD. S.T. EVERY ELT. MORI CONTR. IS A $\mathbb{P}^1$-FIBRATION $\implies X = G/B$ FOR SOME S-SIMPLE G.

[CAN BE STRENGTHENED]

GEOMETRIC CHARACTERIZATION OF FLAGS.

RIGIDITY $\rightsquigarrow$ LOCAL (CONTROLLED BY E.G. $H^1(TX)$)

GLOBAL (HARD)

$\mathcal{X} \xrightarrow[\text{PROPER}]{\text{SMOOTH}} \Delta \quad 0 \in \Delta \ni t$ $\quad$ 0 SPECIAL t GENERAL

C^∞ EHRESMANN THM $\implies X_0 \underset{C^\infty}{\cong} X_t \quad \mathcal{X} \underset{C^\infty}{\cong} \Delta \times X_0$

SIMPLIFYING ASS: $\mathcal{X}$ IS PROJECTIVE / Δ

EXAMPLE (FUNDAMENTAL): HIRZEBRUCH SURFACE

$X_t = \mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \quad X_0 \cong \mathbb{F}_{2e}$ - EVEN HIRZ. SURFACE

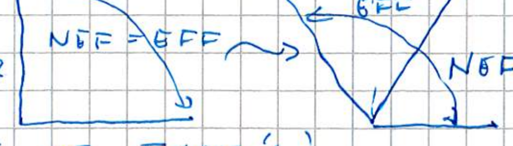
$$0 \to \mathcal{O}(-e) \to \mathcal{E} \to \mathcal{O}(e) \to 0 \qquad \Delta \cong \mathrm{Ext}^1(\mathcal{O}(e), \mathcal{O}(-e))$$

$$\mathcal{X} = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \times \Delta \to \Delta$$

EXPLANATION: NEW CYCLE

LOOK AT CONES

IN $N_1 = {}^{1\text{-CYCLES}}/_{\equiv} \otimes \mathbb{R}$ $\quad N^1 = {}^{\mathrm{Pic}}/_{\equiv} \otimes \mathbb{R}$



HOWEVER IF $-K_{\mathcal{X}}$ IS AMPLE (FAMILY OF FANO'S) THEN CONES ARE RIGID

THEOREM II LET $\mathcal{X} \to \Delta$ BE A FAMILY OF SMOOTH FANO VARIETIES. IF $X_t \cong G/B$ THEN $X_0 \cong G/B$

[RIGIDITY THM FOR COMPLETE FLAGS]

REMARKS ① CASE G/P_{max} - GRASSMANNIAN WAS STUDIED BY HWANG AND MOK; THEY ARE USUALLY RIGID (BUT NOT ALWAYS)

② EXAMPLE: ω - SYMPLECTIC FORM ON V^{2n} ⇝ CONTACT STRUCTURE ON $\mathbb{P}^{2n-1} = \mathbb{P}(V)$

$$0 \to F \to T\mathbb{P}^{2n-1}(-1) \to L \to 0$$

$$\underset{\substack{\uparrow \\ \text{CONTACT FORM}}}{0 \to \mathcal{O}} \to \Omega(2) \to F^*(2) \to 0$$

$\mathbb{P}^{1}$ BUNDLE, $\mathbb{P}^1$

$\mathbb{P}^{2n-1} \leftarrow \mathbb{P}(\Omega(2)) \to \mathrm{Gross}(2,\ldots)$

DEGENERATES

BAD NEWS!

$\mathbb{P}(\mathcal{O} \oplus N^*(2))$

$\mathbb{P}^{2n-1} \leftarrow \mathbb{P}(F^*(2)) \to \mathrm{Gross}$

IDEA OF PROOF OF THM II:

USE THM I ⇝ PROVE THAT ANY $\mathbb{P}^1$-BUNDLE STRUCTURE ON $G/P = X_t$ DEGENERATES TO $\mathbb{P}^1$ BUNDLE ON X_0

EXAMPLE ABOVE ⇝ IN GENERAL THIS IS FALSE!

BY THEOREM OF FUJITA NEED TO PROVE $\varphi_0 : X_0 \to Y_0$ EQUIDIM

FANO MORI CONTR. $X_t \hookrightarrow \mathcal{X}$ RELATIVE CONTRACTION

$Y_t \hookrightarrow \mathcal{Y}$ ← FAMILY OF POLARIZED RING VARIETIES

THIS CAN BE CONTR. BY COHOMOLOGY BECAUSE FIBER TRANSVERSAL TO $\mathbb{P}^1_t$

$$H^*(X_t) \cong H^*(\mathcal{X}) \cong H^*(X_0)$$

$$H^*(Y_t) \xleftarrow{\text{RESTRICTION}} H^*(\mathcal{Y}) \cong H^*(Y_0)$$

IF ISOMORPHISM THEN WE ARE DONE

BECAUSE $H^*(X_t)$ FREE $H^*(Y_t)$ MODULE

MOREOVER WE KNOW $H^i(Y_t) \cong H^i(Y_0)$ FOR $i = 0, 1, 2$.

SO IT IS ENOUGH TO PROVE $H^*(Y_t)$ GENERATED IN ≤ 2 GRADATION.

THEOREM (BOREL BERNSTEIN GELFAND^2) S POLYNOMIAL RING ON CHARACT. OF THE MAX TORUS OF G, W_P - WEYL GROUP GENERATED BY REFLECTIONS ASS. TO $P \subset G$, $W_P \subset W$ = WEYL

$$H^{2*}(G/P) = S^{W_P} / (S_+^{W}) \leftarrow \text{IDEAL GEN BY POSITIVE DEGREE INV.}$$

IDEA OF PF OF THM I.

FIRST IDENTIFY G, THEN USE REVERSE ENGINEERING TO PROVE $X \cong G/P$

STEP 1:

LEMMA $\varphi : X \to Y$ $\mathbb{P}^1$ BUNDLE / FIBRATION

$K := K_{X/Y}$, Γ FIBER. TAKE $D \in \operatorname{Pic} X$

IF $l = -1$ THEN $H^i(\mathcal{O}(D)) \Leftarrow [\, l := D.\Gamma$

OTHERWISE $H^i(X, \mathcal{O}(D)) = H^{i + \operatorname{sgn}(l+1)}(X, \mathcal{O}(D + (l+1)K))$

NOW GIVEN X AS IN THM 1, FOR EVERY

$\varphi_i : X \to Y_i$ ELEMENTARY CONTRACTION ($\mathbb{P}^1$-BUNDLE)
DEFINE $\sigma_i : N^1(X) \to N^1(X)$

$$\sigma_i(D) = D + [D.\Gamma_i + 1] . K_i$$

THIS IS AFFINE REFLECTION ~~EQ~~ UP TO SHIFT BY K_i
REFLECTION WRT THE FACE OF $\operatorname{NEF}(X)$
ASS. TO φ_i, I.E. $\Gamma_i^{\perp}$

NOTE: BY KODAIRA VANISHING $H^i(\mathcal{O}(D)) = 0$, $i > 0$
FOR $D \in \operatorname{NEF}(X)$

DEFINE $W = \langle \sigma_i \rangle$ REFLECTION GP.

(1) IT IS FINITE (BECAUSE LENGTH $\leq \dim X$ OF ELT.)

(2) IT PRESERVES $\pm \chi$ HILBERT POLYNOMIAL ON $N^1(X)$

(3) IT PRESERVES LATTICE $\operatorname{PIC}(X)$

* BY CLASS OF ~~FIN~~ REFLECTION GPS IT IS WEYL GP FOR SOME S-SIMPLE G $(W = W(G))$

* $\exists$ G/B HAS THE SAME COHOMOLOGY ON PIC

I.E. $N^1_{\mathbb{Q}}(X) \simeq N^1(G/B)$ $\quad H^i(X, D) \simeq H^i(G/B, D')$

$D \to D'$

STEP 2: REVERSE ENGINEERING

USE BOTT-SAMELSON VAN | GIVEN A SEQUENCE OF CONTRACTIONS $\varphi_i : X \to Y_i$ DEFINE. A WORD w

Z_w is proj. of $\mathcal{J}$ over Z_{w_i}

$$0 \to \varphi_w^* K_i \to \mathcal{J} \to \mathcal{O} \to 0$$

$$\begin{tikzcd} & Z_{w_i} \arrow[dr] & \\ Z_w \arrow[ur, bend left] \arrow[rr, "\Phi_w"] \arrow[drr] & & X \arrow[d, "\varphi_i"] \\ & & Y_i \end{tikzcd}$$

CONCLUSION: IN MOST CASES Z_w CAN BE CONTROLLED BY CALCULATING COHOMOLOGY

* CASES SIMPLY LACED: A, D, E : EVERY Z_w IS DETERMINED UNIQUELY
* CASES B & C : THERE EXISTS A SEQUENCE OF Z_w's UP TO THE MAXIMAL ONE WHICH IS UNIQUE
* CASE G_2, F_4 OTHER METHODS NEEDED

FINAL STEP

$$\begin{tikzcd} & Z_{MAX} \arrow[dl] \arrow[dr, "\text{BIRATIONAL}"] & \\ X & & G/B \arrow[ll] \end{tikzcd}$$

FINITE $\Leftrightarrow$ SAME CANONICAL $\Rightarrow$ ISOMORPHIC