Infinitesimal Automorphisms of Cubic Hypersurfaces and Projective Legendrian Manifolds

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Degeneracy of quadratic forms

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- q is degenerate $\Leftrightarrow$ Q is a cone $\Leftrightarrow$ Q is singular.

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  for all $u, v, w \in V$.
- There are (at least) three different degeneracy conditions:

(D1) $\exists u \neq 0, \forall v, \forall w \in V,\ f(u, v, w) = 0.$

(D2) $\forall u, \exists v \neq 0, \forall w \in V,\ f(u, v, w) = 0.$

(D3) $\exists u \neq 0, \forall w \in V,\ f(u, u, w) = 0.$

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- $\mathrm{Sing}(\widehat{Y}) = \{uu \in V,\ f_{uu} = 0\}$.
- The rational map $\Phi : \mathbb{P}V \dashrightarrow \mathbb{P}V^\vee$ defined by
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- (D2) $\Leftrightarrow$ The polar map Φ is not dominant.

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$$\{\text{cubics that are not cones}\}$$
$$\bigcup$$
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$$\{\text{nonsingular cubics}\}$$

Infinitesimal automorphisms

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$$\mathrm{aut}(\widehat{Y}) = \mathrm{aut}(Y) + \mathbb{C}\mathrm{Id}_V \;\subset \mathrm{End}(V)$$

- $\varphi \in \mathrm{aut}(\widehat{Y})$ satisfies

$$f(\varphi(u), v, w) + f(u, \varphi(v), w) + f(u, v, \varphi(w)) = \chi(\varphi) f(u, v, w)$$

for some character $\chi : \mathrm{aut}(\widehat{Y}) \to \mathbb{C}$.

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- Y is a cone $\Rightarrow \ \mathrm{aut}(Y) \neq 0$.
- Y is nonsingular $\Rightarrow \ \mathrm{aut}(Y) = 0$.
- **Question** If Y has nonzero Hessian, is $\mathrm{aut}(Y)$ is small ??

Secants of Severi varieties

- Four Severi varieties:

$$v_2(\mathbb{P}^2) \subset \mathbb{P}^5,\ \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8,\ \mathrm{Gr}(2,6) \subset \mathbb{P}^{14},\ \mathbb{OP}^2 \subset \mathbb{P}^{26}.$$

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- How to interpret unusually large?

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- For a projective variety $Z \subset \mathbb{P}V$ and the Lie algebra $\mathrm{aut}(\widehat{Z}) \subset \mathrm{End}(V)$, we say that $A \in \mathrm{Hom}(\mathrm{Sym}^2 V, V)$ is a prolongation of $\mathrm{aut}(\widehat{Z})$ if $A_u \in \mathrm{aut}(\widehat{Z})$ for all $u \in V$.

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- An element of $\mathrm{aut}(\widehat{Z})^{(1)}$ can be considered as a higher order automorphism of $Z \subset \mathbb{P}V$.

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- Denote by Ξ^c_Y the space of prolongations of polar type with weight c.

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- If $Y \subset \mathbb{P}V$ is the secant of a Severi variety $S \subset \mathbb{P}V$, then

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- In fact, for $A \in \mathrm{aut}(\widehat{Y})^{(1)}$ and all $u, v \in V$,

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where $h^A \in \mathrm{Hom}(V^\vee, V)$ an isomorphism which idetifies S with the projective dual of Y.

Main Questions

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Main Theorem

Theorem (– 2016)

Let Y be an irreducible cubic hypersurface satisfying

(a) *the reduced singular locus $\mathrm{Sing}(Y)$ is nonsingular, and*

(b) *Y has nonzero Hessian.*

If $\Xi_Y^c \neq 0$ for some $c \neq \frac{1}{4}$, then Y is the secant of a Severi variety.

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- If we replace (a) by the stronger assumption that the singular locus scheme (defined by the ideal $\{f_{uu}, u \in V\}$) is nonsingular, then the proof becomes much easier. But the result will be less useful.

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Theorem (Characterization of Severi Varieties)

Let $S \subset \mathbb{P}V$ be a nondegenerate nonsingular variety such that $\mathrm{Sec}(S)$ is a hypersurface. If $\mathrm{aut}(\widehat{S})^{(1)} \neq 0$, then S is a Severi variety.

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This characterization of Severi varieties is an easy consequence of the following classification result.

Prolongation of nonsingular varieties

Theorem (Baohua Fu –, 2016)

Let $S \subset \mathbb{P}V$ be a nondegnerate nonsingular variety with $\mathrm{aut}(\widehat{S})^{(1)} \neq 0$. Then S is one of the following:

(i) *VMRT of irreducible Hermitian symmetric spaces*

(ii) *VMRT of symplectic and odd-symplectic Grassmannians*

(iii) *a nonsingular linear section of $\mathrm{Gr}(2, \mathbb{C}^5) \subset \mathbb{P}^9$, of dimension 4 or 5*

(iv) *a (special) nonsingular linear section of the Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$, of dimension 7, 8 or 9*

(v) *certain biregular projections of varieties in (i) or (ii).*

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- Proof by classifying certain birational transformation of $\mathbb{P}^n$ to Fano manifolds of Picard number 1, called 'Special birational transforms of type (2, 1)'.

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- Assuming $Y \neq \mathrm{Sec}(\mathrm{Sing}(Y))$, one shows that any $A \in \mathrm{aut}(\widehat{Y})^{(1)}$ has the form

$$A_{uu} = \frac{1}{4}\chi^A(u)v + \frac{1}{4}\chi^A(v)u + \text{(Gauss map part)},$$

leading to a contradiction if $c \neq \frac{1}{4}$.

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- A projective variety $Z \subset \mathbb{P}W$ is a projective Legendrian variety if its affine cone $\widehat{Z} \subset W$ is a Lagrangian cone. If Z is furthermore nonsingular, we call it a projective Legendrian manifold.

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Ex2 For each simple Lie algebra $\mathfrak{g}$, different from $\mathfrak{sl}$ and $\mathfrak{sp}$, there exists a homogeneous Legendrian manifold $Z^{\mathfrak{g}} \subset \mathbb{P}W^{\mathfrak{g}}$ called the subadjoint variety of $\mathfrak{g}$, in a suitable symplectic vector space $W^{\mathfrak{g}}$.

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Ex3 (Segre/Bryant) Any nonsingular curve can be embedded in $\mathbb{P}^3$ as a Legendrian curve.

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Ex3 (Segre/Bryant) Any nonsingular curve can be embedded in $\mathbb{P}^3$ as a Legendrian curve.

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Examples of projective Legendrian manifolds

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Ex5 (Buczynski) There are many quasi-homogeneous projective Legendrian manifolds which are not homogeneous.

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- This cubic form $f_z(Z)$ is called the third fundamental form of Z at z.

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- The third fundamental forms of $Z^{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ are exactly the secants of the four Severi varieties.

Characterization of subadjoint varieties by third fundamental forms

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Theorem (–)

For a projective Legendrian manifold $Z \subset \mathbb{P}W$, $\dim Z \geq 2$, if $f_Z(Z)$ at a general point $z \in Z$ is isomorphic to $f_x(Z^{\mathfrak{g}})$ at a point $x \in Z^{\mathfrak{g}}$, then Z is isomorphic to the subadjoint variety $Z^{\mathfrak{g}}$.

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- For $\mathfrak{g} \neq \mathfrak{so}$, this was proved by [– Yamaguchi, 2001].
- For $\mathfrak{g} = \mathfrak{so}$, the proof uses deformation theory and Pirio-Russo's work on projective manifolds 3-connected by twisted cubics.

Cubic hypersurface defined by the third fundamental form

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For a nondegenerate projective Legendrian manifold $Z \subset \mathbb{P}W$*, let* $Y_z \subset \mathbb{P}T_z(Z)$ *be the cubic hypersurface defined by the third fundamental form* $f_z(Z) \in \mathrm{Sym}^3 T_z^\vee(Z)$ *at a general point* $z \in Z$*. Then*

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- (b) can be checked by studying the dual variety of Z.

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Theorem (–)

Let $Z \subset \mathbb{P}W$ be a nondegenrate projective Legendrian manifold and let $z \in Z$ be a point. Let

$$\rho_z : \mathrm{aut}(z \in Z) \to \mathrm{End}(T_z(Z))$$

be the isotropy representation of the isotropy subalgebra $\mathrm{aut}(z \in Z) \subset \mathrm{aut}(Z)$.

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Idea of Proof:

- The 2nd order Taylor expansion of an element of $\mathrm{Ker}(\rho_z)$ is an element of $\Xi_{Y_z}^{1/2}$.

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Idea of Proof:

- The 2nd order Taylor expansion of an element of $\mathrm{Ker}(\rho_z)$ is an element of $\Xi^{1/2}_{Y_z}$.
- Main Theorem implies that the third fundamental form at z is isomorphic to that of $Z^{\mathfrak{g}}$ for some $\mathfrak{g}$.

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- For a projective variety $Z \subset \mathbb{P}W$ and the Lie algebra $\mathrm{aut}(Z) \subset \mathfrak{sl}(W)$, an element $\beta : \wedge^2 W \to \mathrm{aut}(Z)$ is called a Bianchi tensor of $Z \subset \mathbb{P}W$ if

$$\beta(u, v)w + \beta(v, w)u + \beta(w, u)v = 0$$

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- The subadjoint variety $Z^{\mathfrak{g}} \subset \mathbb{P}W^{\mathfrak{g}}$ has no nonzero Bianchi tensor ([Yamgauchi, 1992]).

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Idea of Proof: A Bianchi tensor induces an element of $\Xi^0_{Y_z}$ at a general point $z \in Z$, which vanishes by Main Theorem.

Thank you very much !!