$\mathbb{Q}$-Homology Projective Planes

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Outline

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A minimal resolution of S has $p_g = q = 0$.

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Problem (Kollár)

Classify all $\mathbb{Q}$-homology $\mathbb{CP}^2$'s with quotient singularities.

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With DongSeon Hwang, we classified S with $|Sing(S)| = 5$.

Theorem (D.Hwang-Keum, J. Alg. Geom. 2011)

Let S be a $\mathbb{Q}$-homology $\mathbb{CP}^2$ with quotient singularities. Then $|Sing(S)| \leq 4$ except the following case:

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Remark.
1. $\mathbb{Q}$-homology $\mathbb{CP}^2$ with rational singularities may have an arbitrary number of singularities, no bound.
2. In char 2 there is an example with 7 A_1-singularities.

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Theorem (The weak oBMY inequality – Keel-Mckernan, M.AMS 1999)

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$$0 \leq e_{orb}(S).$$

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- Finally, construct an example of type $3A_1 \oplus 2A_3$.
  Consider an Enriques surface S' with elliptic fibration of type $I_2 + I_2 + I_4 + I_4$ and a 2-section intersecting only one component of each fibre.

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Definition

A $\mathcal{C}^\infty$-action of $\mathbf{S}^1$ on $\mathbf{S}^{2n-1}$

$$\mathbf{S}^1 \times \mathbf{S}^{2n-1} \to \mathbf{S}^{2n-1}$$

is called a pseudofree $\mathbf{S}^1$-action on $\mathbf{S}^{2n-1}$ if it is free except for finitely many orbits (whose isotropy groups $\mathbb{Z}/a_1, \dots, \mathbb{Z}/a_k$ have pairwise prime orders).

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Example

Linear actions

$\mathbf{S}^{2n-1} = \{(z_1, z_2, ..., z_n) : |z_1|^2 + |z_2|^2 + ... + |z_n|^2 = 1\} \subset \mathbb{C}^n$,
$\mathbf{S}^1 = \{\lambda : |\lambda| = 1\} \subset \mathbb{C}$.

$$\mathbf{S}^1 \times \mathbf{S}^{2n-1} \to \mathbf{S}^{2n-1}$$

$$(\lambda, (z_1, z_2, ..., z_n)) \to (\lambda^{a_1} z_1, \lambda^{a_2} z_2, ..., \lambda^{a_n} z_n),$$

$a_1, ..., a_n$ pairwise prime.

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$a_1, ..., a_n$ pairwise prime.

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Furthermore, Σ *is diffeomorphic to* $\mathbf{S}^5$ *iff* $\pi_1(M) = 1$.

Outline

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These examples obtained from the classification of surface quotient singularities [E. Brieskorn, Invent. Math. 1968].

Example (coming from Brieskorn's classification)

$I_m \subset GL(2, \mathbb{C})$ the $2m$-ary icosahedral group $I_m = \mathbb{Z}_{2m}.\mathcal{A}_5$.

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Call these Brieskorn quotients.

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Remark. If $H_1(S^0, \mathbb{Z}) = 0$, K_S cannot be numerically trivial.

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Question: Are there such a surface S with $|Sing(S)| = 4$?

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- Keel and Mckernan (Mem. AMS 1999),
- Kollár (Pure Appl. Math. Q. 2008) — an infinite series of examples with $|Sing(S)| = 2$.
- D. Hwang and Keum (Proc. AMS 2012) — infinite series of examples with $|Sing(S)| = 1, 3$.

Question: Are there such a surface S with $|Sing(S)| = 4$?
Are there any $\mathbb{Q}$-homology $\mathbb{CP}^2$ which is a rational surface S with K_S ample and with $|Sing(S)| = 4$?

Cascade structure on $\mathbb{Q}$-homology $\mathbb{CP}^2$'s

A $\mathbb{Q}$-homology $\mathbb{CP}^2$ is obtained from a basic surface with certain configuration of curves, by blow-ups and downs.
In the case where $-K_X$ ample, such basic surfaces have been classified by D. Hwang.

Outline

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Let R be the singularity type or the corresponding root sublattice of the cohomology lattice of S', the minimal resolution of S.
Since S is Gorenstein, rank(R) is bounded.

$$1 + rank(R) = b_2(S') = 10 - K_{S'}^2 = 10 - K_S^2 \leq 10,$$

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with equality iff $K_S \equiv 0$ iff S' is an Enriques surface.

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With D. Hwang and H. Ohashi, have classified all possible singularity types of Gorenstein $\mathbb{Q}$-homology $\mathbb{CP}^2$'s.
There are 58 types.

Theorem (Hwang-Keum-Ohashi, Sci. China Math. 2015)

The singularity type R of a Gorenstein $\mathbb{Q}$-homology $\mathbb{CP}^2$ is one of the following:

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- *27 types for K_S $\pm$ample:*
  A_8, A_7, D_8, E_8, E_7, E_6, D_5, A_4, A_1;
  $A_7 \oplus A_1$, $A_5 \oplus A_2$, $A_5 \oplus A_1$, $2A_4$, $A_2 \oplus A_1$, $D_6 \oplus A_1$,
  $D_5 \oplus A_3$, $2D_4$, $E_7 \oplus A_1$, $E_6 \oplus A_2$;
  $A_5 \oplus A_2 \oplus A_1$, $2A_3 \oplus A_1$, $A_3 \oplus 2A_1$, $3A_2$, $D_6 \oplus 2A_1$;
  $2A_3 \oplus 2A_1$, $4A_2$, $D_4 \oplus 3A_1$,
- *31 types for K_S numerically trivial:*
  A_9, D_9;
  $A_8 \oplus A_1$, $A_7 \oplus A_2$, $A_5 \oplus A_4$, $D_8 \oplus A_1$, $D_6 \oplus A_3$, $D_5 \oplus A_4$,
  $D_5 \oplus D_4$, $E_8 \oplus A_1$, $E_7 \oplus A_2$, $E_6 \oplus A_3$;
  $A_7 \oplus 2A_1$, $A_6 \oplus A_2 \oplus A_1$, $A_5 \oplus A_3 \oplus A_1$, $A_5 \oplus 2A_2$,
  $2A_4 \oplus A_1$, $3A_3$, $D_7 \oplus 2A_1$, $D_6 \oplus A_2 \oplus A_1$, $D_5 \oplus A_3 \oplus A_1$,
  $2D_4 \oplus A_1$, $E_7 \oplus 2A_1$, $E_6 \oplus A_2 \oplus A_1$;
  $A_5 \oplus A_2 \oplus 2A_1$, $A_4 \oplus A_3 \oplus 2A_1$, $2A_3 \oplus A_2 \oplus A_1$, $A_3 \oplus 3A_2$,
  $D_6 \oplus 3A_1$, $D_4 \oplus A_3 \oplus 2A_1$;
  $2A_3 \oplus 3A_1$.

The 27 types with $-K_S$ ample were classified by Furushima(1986), Miyanishi and Zhang(1988), Ye(2002). Our method uses only lattice theory, different from theirs.

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Currently, M. Schütt constucts, for each of the 31 types, an explicit 1-dimensional family of Enriques surfaces.

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The first component corresponds to Enriques surfaces of Kondō type I,
the second to Enriques surfaces $Km(E \times E)/i_L$ where i_L is the Lieberman involution.