Iitaka conjecture and abundance for 3-folds in $\mathrm{char} = p > 5$

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Classification of varieties

A task of algebraic geometry is **to classify smooth projective varieties**.

$\dim X = 1$**:**

$g = h^0(X, \omega_X) \rightsquigarrow \mathcal{M}_g$: moduli of curves of genus g.

$\dim X = 2$**:**

(1) **Difficulty**: there are too many surfaces X_n birational to X, i.e., $\exists$ open sets $U_n \subset X_n$ isomorphic to an open set of X.

(2) Reasonable to classify varieties up to birational equivalence.

(3) Find a good model from a birationally equivalent class in a standard way.

(4) **Strategy**: choosing the "minimal" variety X, which admits no blowing down maps.

Minimal model theory in dimension two

Solution in dimension 2

For a surface X, one can go a minimal model program (MMP for short)

$$\text{blowing down } X \to X_1 \to X_2 \to \ldots \to X^m = \bar{X}$$

to get a smooth birational model $\bar{X}$ such that either
(1) $\bar{X} \cong \mathbb{P}^2$ or is ruled by rational curves, or
(2) $K_{\bar{X}}$ is nef and is semi-ample.
In the latter case $\bar{X}$ is called the **minimal model** of X.

Kodaira dimension

Kodaira dimension is an invariant to refine the classification.

- Kodaira dimension: $\kappa(X) := \max_{m>0} \dim \phi_{|mK_X|}(X)$, which can be $-\infty, 0, 1, \cdots, \dim X$.
- For surfaces, in case (1), $\kappa = -\infty$.
- In case (2), $\kappa = 0, 1, 2$, and the case $\kappa = 0$ is completely classified; when $\kappa = 0$ Iitaka fibration of $\bar{X}$ is an elliptic fibration; when $\kappa = 2 = \dim X$ we say X is *of general type*.
- A divisor D is said to be semi-ample, if for sufficiently divisible $N > 0$, $|ND|$ has no base point.
- In Case (2), $K_{\bar{X}}$ is semi-ample. We say "**abundance**" holds for surfaces.

Minimal model program

$dimX \geq 3$**:** similar story happens.
From X, after a sequence of birational modification (divisorial contraction, flips), the so-called "MMP" one expects to get a "minimal" variety $\bar{X}$ such that either

- $\bar{X}$ is a Mori fiber space, or
- $K_{\bar{X}}$ is nef.

Two central problems of MMP include

- Existence of minimal models in dimension n (E_n): Can MMP run and terminate?
- Abundance in dimension n (A_n): if K_X is nef, is K_X semi-ample?

Log-minimal model program

To study MMP by induction on dimensions, varieties with boundary (X, B) should be considered. Analogous theory is called **log MMP**.

Central problems of log-MMP include

- Existence of log minimal models in dimension n (log E_n)
- Abundance in dimension n for lc minimal models (log A_n)

Remark: Mild singularities, say, singularities below should be permitted in higher dimension to make MMP run. For a pair (X, B), taking a log smooth resolution $f : Y \to X$ and writing that

$$K_Y = f^*(K_X + B) + \sum_i a_i E_i,$$

if $a_i \geq -1$ (> -1), we say (X, B) is **log-canonical (Kawamata log terminal)**.

Progresses in characteristic 0 of MMP

Over complex numbers, remarkable progresses were made by many brilliant mathematicians: Mori, Kawamata, Shokurov, Reid, Kollar, Viehweg, BCHM, Fujino, Gongyo....
The following has been proved

- log E_n and log A_n when B is big or $K_X + B$ is big [BCHM, 2010];
- log E_3 and log A_3 [Miyaoka, Kawamata, KMM, 1990s].

Progresses in characteristic p

In dimension 2,

- existence of minimal models and abundance are implied by Bombieri-Mumford's classification [BM, 1970s],
- similar results are true for (semi-)log canonical surfaces [Tanaka, 2014-15].

In dimension 3 and char $p > 5$,

- existence of minimal models of klt pairs (X, B) [Hacon, Xu, Birkar, 2013-15];
- abundance for klt pairs when either $K_X + B$ or B is big [Birkar and Xu, 2015].

Problem

Prove abundance for a minimal klt pair (X, B) of 3-fold with $\nu(K_X + B) = 0, 1, 2$.

Approach in characteristic zero: $q(X) > 0$

Two completely different methods are employed according to $q(X)$, i.e., whether X has Albanese map.
(1) If $q(X) > 0$, i.e., X has non-trivial Albanese map $a_X : X \to A_X$, one can show abundance by using

Theorem: Kawamama, Viehweg, 1980

Let X be a smooth projective variety over $\mathbb{X}$ of maximal Albanese map. If $\kappa(X) = 0$, then X is birational to an abelian variety.

Iitaka Conjecture

Let $f : X \to Y$ be a fibration between two smooth projective varieties over $\mathbb{C}$, with $\dim X = n$ and $\dim Y = m$. Then

$$C_{n,m} : \kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}).$$

In dimension 3, it is proven by K-V (1980s).

Approach: the case $q(X) = 0$

(2) The case $q(X) = 0$ was treated by [Miyaoka, Kawamata, 1990s].

The approach is as follows

Step 1: Prove the non-vanishing of $H^0(X, nK_X)$. Miyaoka applied Riemann-Roch formula,

$$\begin{aligned}&\chi(Y, \rho^*\mathcal{O}(nK_X)) \\ &= \frac{2n^3 - 3n^2}{12} K_X^3 + \frac{n}{12} K_X \cdot (K_X^2 + \rho_* c_2(Y)) + \chi(\mathcal{O}_X)\end{aligned}$$

the following are needed

- P1 generic positivity of Ω_X (implying $c_2(X) \geq 0$);
- P2 vanishing theorems;
- P3 Donaldson's results on stable bundles: poly-stable vector bundles on a surface with vanishing Chern classes are induced by representation of π_1^{alg};
- P4 the fundamental group of a terminal germ $(X, 0)$ is finite.

Approach continued: the case $q(X) = 0$ and log abundance

Step 2: Prove $\kappa(X) > 0$ if $K_X \not\sim 0$, which implies abundance. The following are needed [Miyaoka and Kawamata, 1980s-90s]

P5 log abundance of surfaces (which has been proven by Tanaka in char $p > 0$)

P6 for case $\nu(K_X) = 1$, infinitesimal deformation of canonical divisors in 3-folds

P7 for case $\nu(K_X) = 2$, log terminal singularities are quotient singularities in codimension two, generic positivity results on orbifolds, vanishing theorems

(3) Log abundance was proved by reducing to Mori fiber spaces [Keel, Matsuki, McKernan, 1990s], where the key point is

P8 canonical bundle formula

Additional difficulties in char $p > 0$

We have additional difficulties in positive characteristics.

P9 For a fibration of smooth varieties $f : X \to Y$, $X_{\bar{\eta}}$ may be singular, in particular it is non-reduced if f is inseparable. So it is much more difficult to show Iitaka conjecture.

P10 For a minimal variety X, X may be uniruled. Then generic positivity of Ω^1_X fails.

Remark:

- We don't worry about P2, P5 by [Tanaka, 2015] and P6 by [Totaro, 2009] in positive characteristics.
- P4 has been studied by [Carvajal-Rojas, Schwede and Tucker, 2015] for F-regular singularities.
- P1 and P8 are widely open and of great importance.

Iitaka conjecture for 3-folds

Theorem: Ejiri, Birkar, Chen, -, 2015-16

Assume $\mathrm{char}\ k = p > 5$. Let (X, B) be a klt 3-folds and $f : X \to Y$ a separable fibration. Then

(1) $C_{3,1}$ is true when

(1.1) $(X_{\overline{\eta}}, B_{\overline{\eta}})$ is klt, $p \nmid \mathrm{ind}(B_{\overline{\eta}})$ and $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + B_{\overline{\eta}}) = 0, 2$; or

(1.2) $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + B_{\overline{\eta}}) = 1$, and $K_{X_{\overline{\eta}}} + B_{\overline{\eta}}$ induces an elliptic fibration.

(2) $C_{3,2}$ is true when

(2.1) $g(X_{\overline{\eta}}) > 0$, or

(2.2) Y is not uni-ruled, K_Y is big, and $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + B_{\overline{\eta}}) = 1$.

Abundance for 3-folds with non-trivial Albanese maps

Theorem: -, 2016

Let X be a $\mathbb{Q}$-factorial, projective, non-uniruled 3-fold, over an algebraically closed field of characteristic $p > 5$. Let B be an effective $\mathbb{Q}$-divisor on X. Assume that
(1) (X, B) is a minimal klt pair; and
(2) the Albanese map $a_X : X \to A_X$ is non-trivial.
Then $K_X + B$ is semi-ample.

Sketch of the proof

(1) Reduce to showing that either $\kappa(K_X + B) \geq 1$ or $K_X + B \sim_{\mathbb{Q}} 0$.
(2) If the Albanese map $a_X : X \to A_X$ is separable, then the Stein factorization of a_X induces a separable fibration $f : X \to Y$. We can prove abundance by use of MMP, subadditivity of Kodaira dimensions, geometry of varieties with $\kappa(X) = 0$.
(3) If the Albanese map $a_X : X \to A_X$ is inseparable, then we have a foliation $\mathcal{F} = \mathcal{L}^{\perp} \subset T_X$ where $\mathcal{L}$ is the saturation of the image of the natural homomorphism $a_X^* \Omega^1_{A_X} \to \Omega^1_X$. By replacing X with $X/\mathcal{F}$ repeatedly, we can finally obtain a variety whose Albanese map is separable, then show that $\kappa(X) \geq 1$ by induction. **Details will be explained below.**

How to treat inseparable maps?

We explain the idea to prove that $\kappa(X) \geq 1$ if the Albanese map $a_X : X \to A_X$ is inseparable, $\dim a_X(X) = \dim X - 1$ and X is not uniruled.

(1) Let $\mathcal{L}$ denote the saturation of the image of the natural homomorphism $a_X^* \Omega^1_{A_X} \to \Omega^1_X$. Then $\mathcal{L}$ is generically globally generated, $\mathrm{rank}\ \mathcal{L} \leq n-2$, and $h^0(X, \mathcal{L}) \geq h^0(A_X, \Omega^1_{A_X}) \geq n-1$, which implies that

$$h^0(X, \det \mathcal{L}) \geq 2.$$

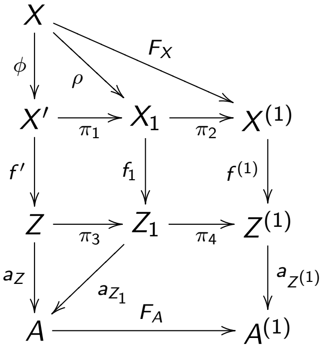
(2) We get a natural foliation $\mathcal{F} = \mathcal{L}^\perp \subset T_X$ of rank ≥ 2 and a quotient map $\rho : X \to X_1 = X/\mathcal{F}$. Then we have

(♣) $\kappa(X) \geq \kappa(X_1)$, and if $\kappa(X_1) = 0$ then $\kappa(X) \geq 1$.

If $a_{X_1} : X_1 \to A$ is separable, then we are done by applying $C_{n,n-1}$.

Continued

(3) Let X' be the normalization of the reduction of $X^{(1)} \times_{Z^{(1)}} Z$. Then



We can show the multiplicity of the geometric fiber of $X_1 \to Z_1$ decrease strictly if a_{X_1} is not inseparable.

Further questions

To prove abundance for 3-folds, the following probably should be concerned according to Miyaoka, Kawamata and KMM

- generic positivity of Ω_X for non-uniruled 3-fold;
- a reasonable canonical bundle formula;
- local fundamental group of klt pairs;
- a method to treat uniruled cases.

On canonical bundle formula

In char $= 0$, for a klt pair (X, B) and a $K_X + B$-trivial fibration $f : X \to Y$, we have a klt pair (Y, B_Y) and the canonical bundle formula (Ambro, 2004):

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).$$

In positive characteristics, Das and Schwede (Ejiri) prove that if (X_η, B_η) is F-split (F-pure) then there exists an effective (pseudo-effective) divisor B_Y on Y such that

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).$$

If not assuming (X_η, B_η) is F-split or F-pure, the results above is not true. One can get an example by considering $P_C(V)$ where V is a semi-stable but not strongly semi-stable. vector bundle of rank two.

Thank you!

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