Higgs de-Rham flows and Applications

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(1) The Start

$k =$ perfect field, char $k = p > 0$

$W_n = W_n(k)$, witt ring of length $n \geq 1$

N. Katz. (1973)

X/W_n : smooth, $X_k = X \times_{W_n} k$

$F_X : X \to X$, flat, $F_X \otimes k =$ absolute Frob.

Then, $\exists$ nat. equ. of cat., $(f \geq 1,\ r \geq 1)$

$$\{\rho \mid \pi_1(X_k) \longrightarrow GL_{r}(W_n \otimes \mathbb{F}_{p^f}),\ \rho \text{ cts}\} \simeq$$

$$\{(E, \Phi) \mid E.\ \text{v.B on } X \text{ of } rk = r,\ \Phi : (F_X^*)^f E \simeq E\}$$

pf: Artin-Schreier. The Φ-invariants give the rep. of π_1.

#

Remark:

(i) $\lceil$ $(n\geq 2)$ Suppose X_k smooth proj curve. Then

(a) $g(X_k) = 0$. Then $\exists F_X : X \to X$

(b) $g(X_k) = 1$ (b1) X_k : ordinary.

$\exists ! F_X : X \to X$, for X/W_n = Serre-Tate lifting

$\nexists F_X$, Otherwise

(b2) X_k : supersingular

$\nexists F_X$ (WRONG! If X_k comes from reduction of CM-ell. curves, then contains the Frob lift!)

(c) $g(X_k) \geq 2$. $\nexists F_X : X \to X$.

Therefore, Katz's correspondence is limited to the smooth affine scheme over W_n if $n \geq 2$.

(Q): How to generalize Katz's corresp. for X/W_n smooth projective?

(ii) $X/\mathbb{C}$: Smooth projective.

Nonabelian Hodge theorem (Uhlenbeck-~~Yau~~, Hitchin, Simpson, Donaldson, Corlette)

$$\{(H,\nabla) \mid \text{flat bundle}\}$$

Riemann-Hilbert $\int\!\!/$ Corellete $\updownarrow$ Simpson, Yang-Mills-Higgs flow

$$\{\rho : \pi_1^{top}(X) \longrightarrow GL_r(\mathbb{C})\} \quad \simeq \{(E,\theta) \mid \text{Higgs bundle, } C_0(E)=r,\ c_1(E)=c_2(E)=0 \text{ semi-stable}\}$$

$$\text{IU} \qquad\qquad \text{IU}$$

$\Updownarrow$ CPVHS

$$\{\rho : \text{Hodge type}\} \quad \simeq \quad \{(E,\theta) \mid \text{graded Higgs bundle } c_1(E)=c_2(E)=0 \text{ semi-stable}, C_0(E)=r\}$$

$$\text{IU} \qquad\qquad \text{IU}$$

$$\{\rho : \text{Hodge-de Rham weight zero}$$

$$\Updownarrow$$

$$\rho : \pi_1 \longrightarrow U_r\} \quad \xrightarrow[\sim]{DUY} \quad \{E \mid E/X : \text{Polystable v.B. } c_0(E)=r,\ c_1(E)=c_2(E)=0\}$$

$$U_r \hookrightarrow GL_r, \quad \pi_1 \searrow GL_r$$

Faltings (2005)

V = complete DVR of mixed char. π = unif.

$K = V[\frac{1}{\pi}]$, $k = V/\pi$, $\hat{\bar{K}} = \mathbb{C}_p$

$X/_{K}$ = smooth proj curve (fix some $A_2(V)$-lifting of X.)

$$\{\text{generalized rep of } \pi_1(X_{\mathbb{C}_p})\} \xrightarrow{\sim} \{(E,\theta)/X_{\mathbb{C}_p} \text{ Higgs bundle}\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\{\rho : \pi_1(X_{\mathbb{C}_p}) \xrightarrow{cts} GL_r(\mathbb{C}_p)\} \underset{?}{\overset{}{\rightleftarrows}} \{(E,\theta)/X_{\mathbb{C}_p} \mid \deg E = 0 \text{ semi-stable}\}$$

Question (Faltings): Does every semi-stable Higgs bundle of $\deg = 0$ come from a $\mathbb{C}_p$-rep of π_1?

Question (Galois actions)

Assume $X_K = X \times_V K$ admit a K-rational point

$$1 \to \pi_1^{geo}(X_{\bar{K}}) \to \pi_1^{arith}(X_K) \to G_K \to 1$$

$\rightsquigarrow \phi : G_K \to Aut(\pi_1^{geo})$.

$\forall [\rho] \in M_{et} = \{\rho : \pi_1^{geo} \xrightarrow{cts} GL_r(\mathbb{C}_p)\}/\simeq$

$\forall \sigma \in G_K$, define

$$\underset{\substack{\shortparallel\\ [\rho^\sigma]}}{[\rho]^\sigma} = [\rho \circ \phi(\sigma^{-1})]$$

Let $[(E,\theta)] \in M_{Dol} = \{(E,\theta)\}/\simeq$ correspond to $[\rho]$.

What is the corresponding Higgs bundle to ρ^σ ?

Note : there is another Galois action :

$$
\begin{tikzcd}
{}^{\sigma}\rho := \pi_1^{geo} \arrow[r, "\mathrm{cts}"] \arrow[rr, bend right, "\mathrm{cts}"'] & GL(\mathbb{C}_p) \arrow[r, "\sigma\text{. isometry}"] & GL(\mathbb{C}_p)
\end{tikzcd}
$$

Then what is the corresponding Higgs bundle to ${}^{\sigma}\rho$?

<u>Conclusion</u>: non-abelian p-adic Hodge theory goes here !

(related to certain dynamics from Galois actions)

(2) Baby example.

X/W = good local model of a Shimura curve of PEL-type (Deligne's strange model)

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr] & X \arrow[d] \\
& W
\end{tikzcd}
$$

universal abelian scheme

$$\mathcal{O}_A$$

$$R^p f_{\mathrm{crys}*}(\Omega_{A/X})$$

$$\| $$

$$(H_{dR}, \nabla^{GM}, F_{hod}, \Phi_{rel}) \in MF^{\nabla}_{[0,1]}(X/W)$$

$$\mathbb{V} \doteq R^1 f_{et*}(\mathbb{Z}_p)_{A_{K_0}} \qquad A_{K_0} = A \times_W W[\tfrac{1}{p}]$$

dual Tate module

$(E,\theta) \triangleq Gr_{F_{hod}}(H_{dR}, \nabla^{GM})$

Assume $W = W(\mathbb{F}_{p^f})$ $f \gg 1$.

Eigen-decomposition:

$$\begin{cases} \mathbb{V} \otimes W = \oplus \mathbb{V}_i, & rk_W(\mathbb{V}_i) = 2 \\ (E,\theta) = \oplus (E_i, \theta_i), & rk_{\mathcal{O}_X} E_i = 2 \end{cases}$$

Work out a theory making the correspondence

$$\mathbb{V}_i \longleftrightarrow (E_i, \theta_i)$$

without reference to the endomorphism str:

Thm (S.-Zuo, 12) X/W: smooth proper, $X_n = X \times_W W_n$. $n \geq 1$.
For any $(H, \nabla, Fil, \Phi) \in MF^{\nabla}_{[0,p-2]}(X/W)$,

$\mathbb{V} \triangleq D(H, \nabla, Fil, \Phi)$ corresponding crystalline rep after Fontaine-Lafaille-Faltings.

$(E,\theta) \triangleq Gr_{Fil}(H, \nabla)$.

Then: $\exists$ 1-1 bijection of sets

$\forall f \geq 1$. $\forall n \geq 1$

$$\{ \mathbb{V}' \subseteq \mathbb{V} \otimes_{\mathbb{Z}_p} W_n(\overline{\mathbb{F}}_{p^f}) \mid \text{sub } \overbrace{\text{crystalline}}^{W_n(\overline{\mathbb{F}}_{p^f})\text{-}} \text{ rep} \}$$

$$\Big\updownarrow$$

$$\{ (E', \theta') \subseteq (E, \theta) \otimes \mathcal{O}_{X_n} \mid (Gr_{Fil} \circ C_n^{-1})^f (E', \theta') = (E', \theta') \}$$

Here the operator $Gr_{Fil} \circ C_n^{-1}$ inductively defined:

$n = 1$: $Gr_{Fil} \circ C_1^{-1}$: $\{$ graded Higgs subbundles of $(E, \theta) \otimes \mathcal{O}_{X_1} \}$ $\circlearrowleft$

C_1^{-1} = inverse Cartier transform of Ogus-Vologodsky

$n = 2$: $Gr_{Fil} \circ C_2^{-1}$: $\{$ graded Higgs subbundles of $(E, \theta) \otimes \mathcal{O}_{X_2}$, which modulo p is periodic $\}$ $\circlearrowleft$

$\vdots$

Remark: (Sheng - Xin - Zuo ~~for~~ the geometric case, Lan - S. - Zuo in general)

(Inverse) Cartier transform of Ogus - Vologodsky = Exponential twisting of the (Classical) Cartier descent.

(3) The notion of Higgs de-Rham flow

X_1/k : Smooth, W_2-liftable (choose & fix a W_2-lifting X_2/W_2 of X_1/k)

A Higgs de-Rham $\underbrace{\text{flow}}_{\text{over } X_1/k}$ is the following diagram

$$
\begin{tikzcd}[column sep=small]
 & (H_0, \nabla_0) \arrow[dr, "Gr_{Fil_0}"] & & (H_1, \nabla_1) \arrow[dr, "Gr_{Fil_1}"] & \\
(E, \theta) = (E_0, \theta_0) \arrow[ur, "C_1^{-1}"] & & (E_1, \theta_1) \arrow[ur, "C_1^{-1}"] & & \cdots
\end{tikzcd}
$$

- (E, θ) = nilpotent Higgs module over X_1/k of exp $\leq p-1$

- Fil_i on $(H_i, \nabla_i) = C_1^{-1}(E_{i-1}, \theta_{i-1})$ is a Griffiths-trans. filtration of level $\leq p-1$

- $(E_i, \theta_i) = Gr_{Fil_{i-1}}(H_{i-1}, \nabla_{i-1})$

Ex:

$$
\begin{tikzcd}[column sep=small]
 & (F^*E, \nabla_{can}) \arrow[dr, "Gr_{Fil_{tr}}"] & & ((F^*)^2E, \nabla_{can}) \arrow[dr] & \\
(E, 0) \arrow[ur, "F^* = C_1^{-1}"] & & (F^*E, 0) \arrow[ur] & & \cdots
\end{tikzcd}
$$

· Preperiodic.

$\exists\ (e, f) \in \mathbb{N}_{\geq 0} \times \mathbb{N}$, s.t

$$(E_0, \theta_0) \nearrow (H_0, \nabla_0) \searrow \cdots \nearrow (H_{e-1}, \nabla_{e-1}) \searrow (E_e, \theta_e) \nearrow \cdots \searrow (E_{e+f}, \theta_{e+f}), \quad (E_{e+f}, \theta_{e+f}) \xrightarrow[\sim]{\varphi} (E_e, \theta_e)$$

If $e=0$, then it is periodic.

· Semi-stable

X_1/k : smooth proj, $H \subset X_1$ ample divisor.

$\forall\ i \geq 0$, (E_i, θ_i) is μ_H - semi-stable.

Thm (Lan-S.-Zuo. 13)

X/W : smooth. $\exists$ equ. of cat ($r \geq 1$, $f \geq 1$)

$\{\rho \mid \pi_1(X_{K_0}) \longrightarrow GL_r(\mathbb{F}_{p^f})$, crystalline$\}$

$\simeq \{(E, \theta, Fil_0, \ldots, Fil_{f-1}, \varphi) \mid f$-periodic Higgs - de Rham flows on $X_1/k\}$

$(E, \theta, Fil_0, \ldots, Fil_{f-1}, \varphi)$ means :

$$
\begin{tikzcd}
 & (H_0, \nabla_0) \arrow[dr, "Gr_{Fil_0}"] & & (H_{f-1}, \nabla_{f-1}) \arrow[dr, "Gr_{Fil_{f-1}}"] & \\
(E, \theta) = (E_0, \theta_0) \arrow[ur, "C_1^{-1}"] & & \cdots \arrow[ur] & & (E_f, \theta_f) \arrow[llll, bend left, "\varphi", "\sim"']
\end{tikzcd}
$$

Inductively, this generalizes to $\forall n \geq 1$,

$$\{\rho \mid \pi_1(X_{K_0}) \longrightarrow GL_r(W_n(\mathbb{F}_{p^f})) \text{ crystalline}\}$$

$$\simeq \{(E, \theta, Fil_0, \ldots, Fil_{f-1}, \varphi) \mid f\text{-Periodic Higgs-de Rham flow on } X_n\}$$

Special case (Assume X/k smooth and W_{n+1}-liftable) :

$\forall\, n \geq 1,\ r \geq 1,\ f \geq 1,$

$$\{\rho \mid \pi_1(X_k) \longrightarrow GL_r(W_n(\mathbb{F}_{p^f})) \text{ cts}\}$$

$$\simeq \{(E, \theta, Fil_{tn}, \ldots, Fil_{tr}, \varphi) \mid f\text{-periodic Higgs de-Rham flow on } X_n \text{ of level} = 0\}$$

$$= \{(E,\varphi) \mid E : \text{v.B on } X_n,$$
$$\varphi : (C_n^{-1})^{f}(E,0) \cong (E,0)\}$$

where, for $\exists F_{X_n} : X_n \longrightarrow X_n$ Frob lifting,

$$(C_n^{-1})(E,0) = (F_{X_n}^{*}E, 0).$$

Prop (S.-Tong, 16)

Notation as above. $k = \mathbb{F}_q$

Suppose $\rho \longleftrightarrow (E,\varphi)$.

Then $\forall \sigma \in G_k$
(i) ${}^{\sigma}\varphi^{\sigma} \longleftrightarrow \sigma^{*}(E,\varphi)$

(ii) ${}^{\sigma}\rho \longleftrightarrow (C_n^{-1}(E), C_n^{-1}(\varphi))$

Thm ($\underbrace{\text{Lan-S.-Yang-Zuo}}_{\text{Lan-S.-Zuo, 13 } (r=2)}$, 14, A. Langer 14)

X/k : smooth proj, W_2-liftable.

(E,θ) : semi-stable Higgs module over X/k.

$rk(E) = r \leq p$.

Then (E,θ) initializes a semi-stable Higgs de-Rham flow.

$r = 2$. Fil = Harder-Narasimhan filtration

r in general; Langer introduced the notion of

Simpson filtration Fil_S

Thus, fix a W_2-lifting X_2 of X_1. get

a stratified self map:

$$Gr_{Fil_S} \circ C_1^{-1} : M_{Dol}(\overline{\mathbb{F}}_p) \circlearrowleft$$

M_{Dol} = moduli of S.S Higgs modules over X_1/k

$$c_0 = r \leq p, \quad c_1 = c_2 = 0.$$

(4) Applications.

(A). Algebraic proof of Bogomolov-Gieseker inequality and Miyaoka-Yau inequality (A. Langer 14)

Remark: The original $\underbrace{\text{proof}}_{\text{transcendental}}$ used Yang-Mills-Higgs metric and Kähler-Einstein metric resp.

This makes the ext. to the quasi-proj case difficulty.

However, the alg proof works over this case well

(Langer, 15)

(B) Hitchin-Simpson Uniformization of p-adic hyperbolic curves with good reductions.

(Lan-S-Yang-Zuo, 16)

(C) Algebraic proof of Kodaira-Saito vanishing thm

(Arapura, 16)

Thm (Arapura)

$X/\bar{\mathbb{F}}_p$ = sm. proj, $D \subset X$ reduced NCD.

Assume (X, D) - W_2 liftable.

Let (E, θ) = nil. semistable log Higgs bundle of exp $\leq p-1$
$c_1(E) = c_2(E) = 0$

L = ample line bundle on X. Then

If $\dim_k(X) + rk(E) \leq p-1$, then

$$\mathbb{H}^i(X, \Omega^*_{Higgs}((E, \theta) \otimes (L, 0))) = 0, \quad i \geq d+1$$

Ω^*_{Higgs} = Higgs complex. (Dolbeault complex by Simpson
Koszul complex by Faltings etc)

Pf: (1) Get a semi-stable Higgs de-Rham flow with Higgs terms of Deligne-Illusie-Raynaud,

$(E_0, \theta_0) \otimes (L, 0)$, $(E_1, \theta_1) \otimes (L^p, 0)$, ...
$\overset{\shortparallel}{(E, \theta)}$

$(E_n, \theta_n) \otimes (L^{p^n}, 0)$, ...

Since $[(E_i, \theta_i)] \in M_{Dol}(\mathbb{F}_q)$

By boundedness (Langer, 04), $\exists (e, f) \in \mathbb{N}_{\geq 0} \times \mathbb{N}$.

s.t
$$[(E_{e+nf}, \theta_{e+nf})] = [(E_e, \theta_e)] \quad \forall n \geq 1$$
(i.e. $\{[(E_i, \theta_i)]\}$ preperiodic)

(2)

$$\begin{tikzcd}[column sep=small]
& C^{-1}(E_i, \theta_i) \otimes (L^{p^{i+1}}, \nabla_{can}) \arrow[dr, "Gr_{Fil}"] & \\
(E_i, \theta_i) \otimes (L^{p^i}, 0) \arrow[ur, "C^{-1}"] & & (E_{i+1}, \theta_{i+1}) \otimes (L^{p^{i+1}}, 0)
\end{tikzcd}$$

Ogus-Vologodsky: Higgs coh. of $(E_i, \theta_i) \otimes (L^{p^i}, 0)$

$\cong$ de-Rham coh of F_* of
$C^{-1}(E_i, \theta_i) \otimes (L^{p^{i+1}}, \nabla_{can})$

$\Longrightarrow$
$$h^*((E_i, \theta_i) \otimes (L^{p^i}, 0)) \leq h^*((E_{i+1}, \theta_{i+1}) \otimes (L^{p^{i+1}}, 0))$$